Inter-School Mathematical Olympiad

ISMO 2023

Time allowed: 2 hours. There are 10 problems, each worth 7 marks. Question are roughly in difficulty order though not necessarily so. *good luck!*

1. Dawn is playing a game with numbers on a whiteboard. She writes the number 20 23 times on the board initially, and at each turn, she chooses two numbers on the whiteboard, a and b, and replaces them with the number $a + b - n$ for some constant n. The game ends when there is only one number left on the board. Find this number in terms of n.

2. If the numbers $p, p + 2$, and $p + 4$ are all prime, show that $p + 6$ is not prime.

3. Let ABC be a triangle with circumcircle ω_1. Let ω_2 be a second circle that is internally tangent to ω_1 at A. Let the intersection of lines AB and AC with ω_2 be D and E respectively. Prove that BC is parallel to DE.

4. Find the smallest integer a such that $\sqrt{a^2 - 2023}$ is an integer.

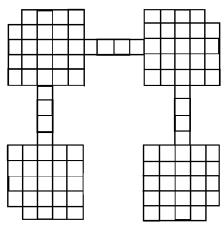
5. Brian and James are playing a game. Initially there are n coins in a pile. Every turn, one of the players (alternating) can remove 3^k (where k is a non-negative integer) coins from the pile. The person who removes the last coin loses. Assuming Brian goes first, for which n can Brian guarantee his victory?

6. Suppose $n = 2^6 \times 3^9 \times 5^{42}$. How many positive divisors of n^2 are not divisors of n?

7. The sequence $a_1, a_2, a_3 \ldots$ is defined by $a_1 = 5$ and $a_2 = 11$, and $a_{n+2} = 2a_{n+1} - 3a_n$ for all integers $n \geq 1$. Prove that a_{2022} is divisible by 11.

8. Consider a point inside a unit square (side length 1) to be brilliant if the sum of the 4 distances from the brilliant point to the 4 vertices of the square is $2\sqrt{2}$. Find the set of brilliant points.

9. Let N be the number of ways of filling this shape with 4-block Tetris shapes (any shape consisting of 4 connected unit squares such that every two either don't intersect, intersect at a single vertex, or share an entire edge. Note that you are not playing Tetris and you can place the pieces anywhere on the grid). Show that there is an integer m such that $(m^2-4m+8)(m^2+4m+8) = N+64$.



10. Let ABC be an acute triangle. Let I be the intersection of the internal bisectors of CAB and ABC. Let D be a point on BC such that ID is perpendicular to BC. Let M be the midpoint of BC. Let X be the reflection of D across M and Y be the reflection of D across I. Let $P \neq D$ be a point such that $MP = MD$ and $ID = IP$.

    (i) Prove that P, X, Y are colinear

    (ii) Prove that A, P, X are colinear

Instructions

1. Do NOT turn over the exam paper until told to do so.
2. You will have 2 hours to complete the examination.
3. Discussion about the exam is prohibited until 11:59pm Monday 3rd April. Anyone found discussing the problems before then is automatically disqualified from the competition.
4. Each question is worth 7 marks.
5. Solutions with explanations are required. A bare answer is usually worth between 0 and 2 marks only.
6. One or two full solutions are likely to earn much more marks than attempts on all 10 problems
7. No electronic devices other than 1 approved non-algebraic function calculator is allowed, graphical instruments (but not protractors) are permitted.
8. Start ALL new questions on a new page
9. All questions about the exam paper must be made within the first 30 minutes. If you have a question, you must write on a separate sheet of A4 paper with the problem number and your question and raise your hand. Wait patiently until an examiner comes to you and collect the sheet of paper. You must not talk during the process
10. If you wish to go toilet during the examination, raise your hand and request to go to toilet via the same format as asking a question. You must not talk during the process.
11. Write on one side of the A4 paper only and clearly label each page with your name, year level, question number, and page number. If you write on both, one of the sides will not be marked.
12. Any violation will invoke a penalty ranging from mark deduction to disqualification.
13. You may leave early, but you must not leave in the last 20 minutes of the exam. If you insist to do so, your score may be invalidated.