2022 Inter-School Mathematical Olympiad

6th-8th April 2022

Time Allowed: 2 hours

Instructions:

- Do NOT turn over until told to do so.
- Do NOT write on the exam paper. It will not be marked.
- A non-algebraic functional calculator may be used; however, other electronic equipment are strictly prohibited. i.e., watches, phones, graphical calculators, laptops, and etc.
- You may not leave your table unless instructed by the examiner.
- There are 10 problems in total, each worth 7 marks. Try work on as many as you can.
- Full solutions (i.e., answer with full justification and working) are required. Generally, an answer by itself will be worth 1 out of 7 points.
- Work on different sheets of paper for each question and label each question clearly.
- Write on one side of each sheet of A4 paper only.
- Hand in all your rough work at the end of the examination.
- You may only ask for clarifications in the first 30 minutes, via raising your hand and waiting patiently.
- Remember, this exam is extremely difficult, solving any one of those problems is an achievement. Good Luck!

Special thank you to Brian Zhao, Grady Kenix, James Xu, Eric Liang, Eric Lee, Sun-woong Kang, Harry Neil, and Michael Ma for making this possible.

1. Find all solutions to the equation $((x+2)^2-7)^2=81$

2. The flag of Nepal is unique as it is not rectangular. Instead, it is made from putting together 2 triangles. The method of making the shape is shown below

    a. Construct a line segment AB
    b. Construct a line segment AC, such that AC is perpendicular to AB and the ratio of length between AB and AC is $3:4$. Let D be on AC such that $AD=AB$. Connect BD.
    c. Let E be a point on BD such that $BE=AB$
    d. Construct FG such that it passes through E, is parallel and equal to AB, and have the point F on AC
    e. Connect CG

Find the area of the flag (Area of polygon $ABEGC$) if $AB=1$.

3. Alex and Brian are playing a game. There is a $\frac{1}{3}$ chance that Alex wins and $\frac{2}{3}$ chance that Brian wins. Alex starts with 100 coins, and Brian with infinitely many. Every time Alex wins, Alex gets half of the amount he currently has from Brian, and loses a quarter every time he loses. Determine whether Alex should play this game or not.

4. Given that $x^2+y=4$, $x,y>0$ find the maximum value of:

    a. x^2y
    b. xy

5. 17 dots with negligible size are put into a 8×8 chessboard. Show that it is always possible to choose 2 dots such that their distance is at most $2\sqrt{2}$

6. Show that for all real numbers a,b,c, $a^2+b^2+c^2+\mathrm{ab}+\mathrm{bc}+\mathrm{ca}\geq 0$

7. The pitch of a sound is determined by its frequency. Call 2 frequencies euphonious if the ratio between them is $2^x:3^x$ or $3^x:2^x$ for some integer x. Find the number of integer frequencies which is euphonious with 22!.
   (for any positive integer n, $n!=n\times(n-1)\times\ldots\times 1$)

8. Aaron and Betty are playing a game consisting of 3 piles of stones containing (a,b,c) where $a\geq b\geq c$ stones with Aaron going first. A move consists of choosing one of the b,c pile and moving a stone from that pile to each of the other piles. A player loses if they cannot make a valid move on their turn. Find all configurations (a,b,c) where Aaron can win.

9. Find all positive integers n such that $\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}$ is an integer.

10. Let ABC be a triangle with incenter I. The foot of the perpendicular from I to BC is D, and the foot of the perpendicular from I to AD is P. Prove that $\angle BPD=\angle DPC$

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