Inter-School Mathematical Olympiad

ISMO Summary 2024

1 Results

1.1 Top Scorers

Senior:

1. Leo Wang (Rangitoto College)
2. John Cai (Rangitoto College)
3. Oscar Prestidge (St Kentigern's College)

Junior:

1. Jason He (Kristin School)
2. Isla Wang (Macleans)
3. Shija Ye (ACG Parnell), Jackie Xu (St Cuthbert's)

1.2 Medal List

First name	Last Name	Division	School	Prize
Leo	Wang	Senior	Rangitoto College	gold
Jason	He	Junior	Kristin	gold
Isla	Wang	Junior	Macleans	gold
Shija	Ye	Junior	ACG	gold
Jackie	Xu	Junior	St Cuthbert	gold
John	Cai	Senior	Rangitoto College	gold
Oscar	Prestidge	Senior	Saint Kentigerns College	gold
Jason	Li	Junior	Macleans	gold
Jerry	Li	Junior	Pinehurst	gold
Benjamin	Chan	Junior	Macleans	gold
Eric	Liu	Junior	Macleans	gold
Gisele	Chong	Junior	Macleans	gold
Alston	Huang	Junior	Rangitoto College	gold
Kyle	Tac	Junior	Rangitoto College	gold
Bella	Chen	Junior	Saint Kentigerns College	gold
Richie	Liu	Junior	Kristin	gold
Vikum	Fernando	Senior	Macleans	silver

Wesley	Lau	Junior	Macleans	silver
Yoonie	Lee	Senior	Saint Kentigerns College	silver
Jason	Tao	Senior	Westlake Boys	silver
Shawn	Zhou	Junior	Westlake Boys	silver
Victor	Tai	Junior	ACG	silver
Jacob	Johnston	Senior	Kristin	silver
Liam	O'Grady	Junior	ACG	silver
Eric	Ni	Junior	Rangitoto College	silver
Qiulin	Yu	Senior	Auckland Grammar	silver
Alex	Zhao	Senior	Westlake Boys	silver
Branden	Li	Junior	ACG	silver
Devin	Xu	Junior	Pinehurst	silver
Yicheng	Wang	Junior	Pinehurst	silver
Derek	Song	Senior	Rangitoto College	silver
Elijah	Tac	Junior	Rangitoto College	silver
Richard	Meng	Senior	Westlake Boys	silver
Harley	Xu	Senior	Macleans	bronze
Felix	Luo	Junior	Rangitoto College	bronze
Eason	Guo	Junior	Kristin	bronze
Mikey	Li	Junior	Kristin	bronze
Orlando	Aguilera	Senior	Rangitoto College	bronze
Selina	Ni	Senior	Rangitoto College	bronze
Anthony	Meng	Junior	Westlake Boys	bronze
Chloe	Seo	Senior	Rangitoto College	bronze
Tony	Guo	Junior	Kristin	bronze
Linghan	Meng	Junior	Pinehurst	bronze
James	Park	Junior	Westlake Boys	bronze
Wilson	Lin	Junior	Kristin	bronze
Hank	Ruan	Junior	ACG	bronze
Joshua	Mu	Junior	ACG	bronze
Enzo	Li	Junior	Long Bay College	bronze
Cael	Sia	Senior	Rangitoto College	bronze
Jayden	Lin	Senior	Rangitoto College	bronze
Isabel	Brennan	Senior	Long Bay College	bronze
Xinyue (Olivia)	Feng	Senior	Rangitoto College	bronze
Alan	Bailey	Senior	Westlake Boys	bronze
Frank	Deng	Junior	Kristin	bronze
Hamish	Patel-Smith	Junior	Kings College	bronze
Finley	Meshouris	Junior	Long Bay College	bronze
Anthony	Hu	Junior	Macleans	bronze
Belinda	Shi	Senior	Macleans	bronze
Borris	Pei	Junior	Macleans	bronze
Jack	Chen	Junior	Pinehurst	bronze

Jiahong	Yu	Senior	Pinehurst	bronze
Daniel	Gao	Senior	Rangitoto College	bronze
Eileena	Bao	Senior	Rangitoto College	bronze
Gordon	Peng	Junior	Rangitoto College	bronze
Larry	Wu	Junior	Rangitoto College	bronze
Siwoo	Jang	Junior	Rangitoto College	bronze
Matthew	Lee	Junior	Kristin	bronze
Sophie	Xu	Junior	Kristin	bronze
James	Barrington	Junior	Kings College	bronze
Selena	Chen	Junior	Long Bay College	bronze
Jimmy	Zhou	Junior	Rangitoto College	bronze
Finch	Ward	Senior	ACG	honourable mention
Cooper	Luck	Senior	Long Bay College	honourable mention
Gavin	Jang	Senior	Rangitoto College	honourable mention
Yeonsu	Na	Senior	Rangitoto College	honourable mention
Kate	Blackett	Senior	Rangitoto College	honourable mention
Kabir	Singh	Senior	ACG	honourable mention
Wen	Yao	Junior	Pinehurst	honourable mention
Josiah	Ong	Junior	ACG	honourable mention
Chace	Chang	Junior	Macleans	honourable mention
Eason	Dong	Senior	Pinehurst	honourable mention
Alan	Li	Junior	Rangitoto College	honourable mention
Tony	Chou	Senior	Auckland Grammar	honourable mention
D'Angelo	Maclaren	Junior	Long Bay College	honourable mention
Jing	Xuan	Senior	Long Bay College	honourable mention
Leo	Huang	Senior	Pinehurst	honourable mention
Sunny	Wu	Junior	Pinehurst	honourable mention
Aria	Hua	Senior	Rangitoto College	honourable mention
Dingding	Mao	Junior	Rangitoto College	honourable mention
Molly	Liu	Senior	Rangitoto College	honourable mention
Tina	Niu	Senior	Rangitoto College	honourable mention
Elaine	Zhou	Junior	St Cuthbert	honourable mention
Ellie	Siu	Junior	St Cuthbert	honourable mention
Lillian	Tang	Junior	St Cuthbert	honourable mention
Ethan	Chong	Senior	Westlake Boys	honourable mention

1.3 Ineligible students

According to the constitution, all students who have been part of the NZMOC January camp are ineligible for monetary or other prizes, but they may still be awarded the respective medal they would have received.

First name	Last Name	Division	School	Prize
Tymon	Mieszkowski	Senior	Long Bay College	gold
George	Zhu	Senior	King's College	gold
Junyi	Guo	Junior	Kristin	gold
Ubeen	Sim	Senior	Westlake Boy's High School	gold
Hanson	Fang	Junior	Kristin	gold
Zhening	Li	Junior	Auckland Grammar	gold
Remi	Geron	Junior	Saint Kentigerns College	gold

1.4 Award Details

Gold: 16 Students, cut off = 28 points
Silver: 17 Students, cut off = 21 points
Bronze: 38 Students, cut off = 13 points
Honourable mention: 24 students, score 7 on at least 1 question

1.5 School Scores[1][2]

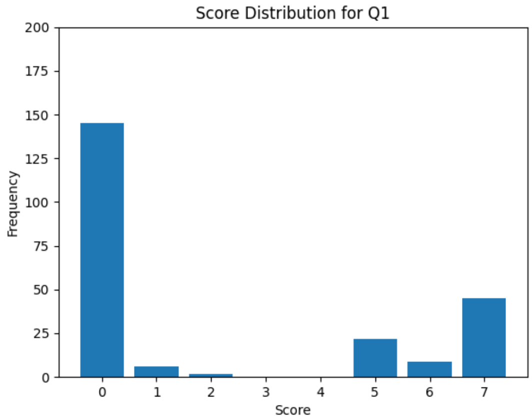
1. Rangitoto College (G,G,G,G,S, 161 points)
2. Macleans (G,G,G,G,G, 152 points)
3. Kristin (G,G,S,B,B, 130 points)
4. ACG Parnell (G,S,S,S,B, 121 points)
5. Westlake Boy's High School (S,S,S,S,B, 114 points)
6. Pinehurst (G,S,S,B,B, 105 points)
7. Long Bay College (B,B,B,B,HM, 70 points)
8. St Cuthbert's College (G,-,HM,HM,HM, 68 points)

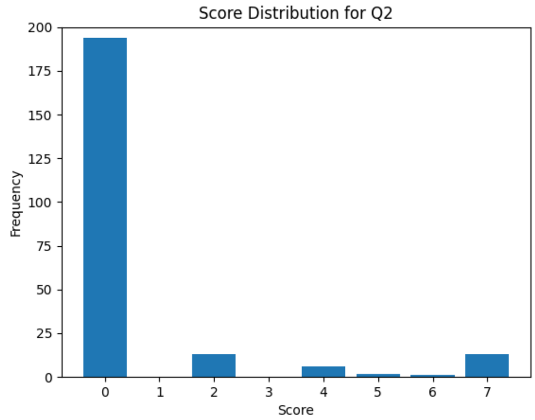
[1]School scores are the sum scores of the top 5 scoring students in each school, G,S,B,HM stands for Gold, Silver, Bronze prizes, and Honourable Mention

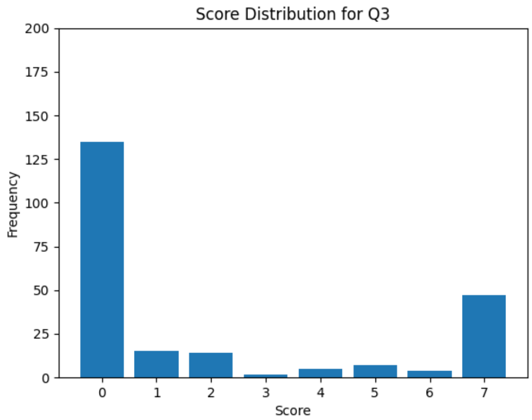
[2]Some schools are not ranked due to lack of students

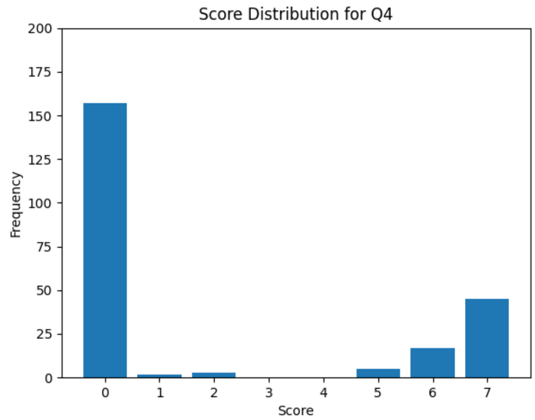
2 Score Distribution

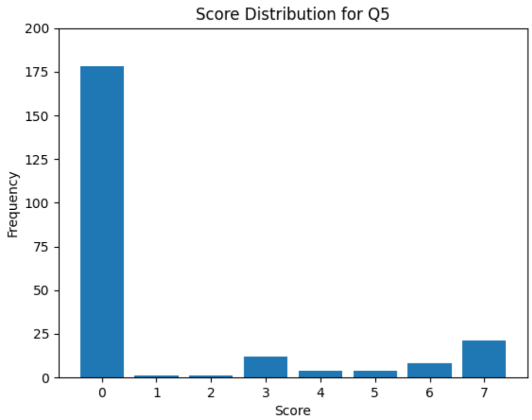
2.1 Score Distribution by Question

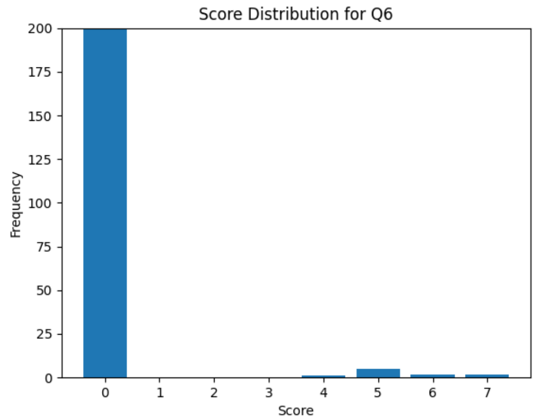


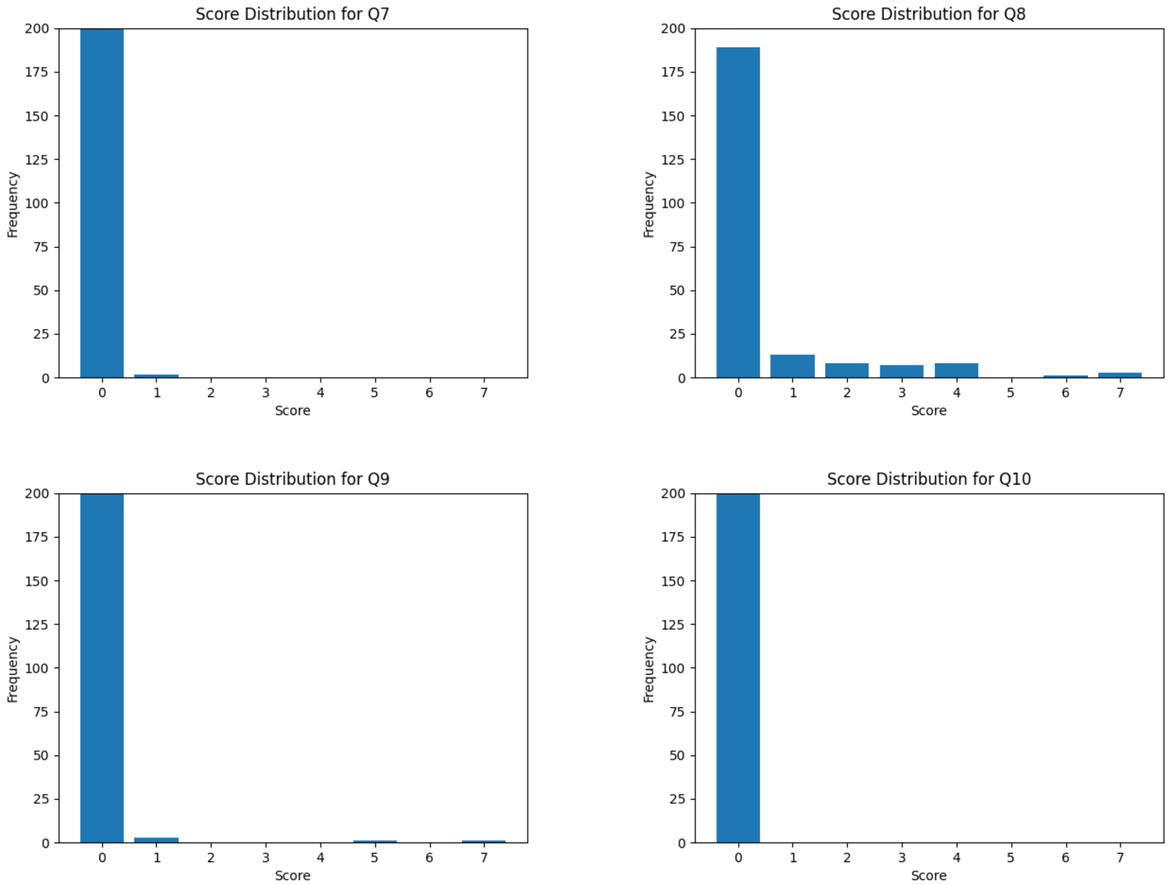




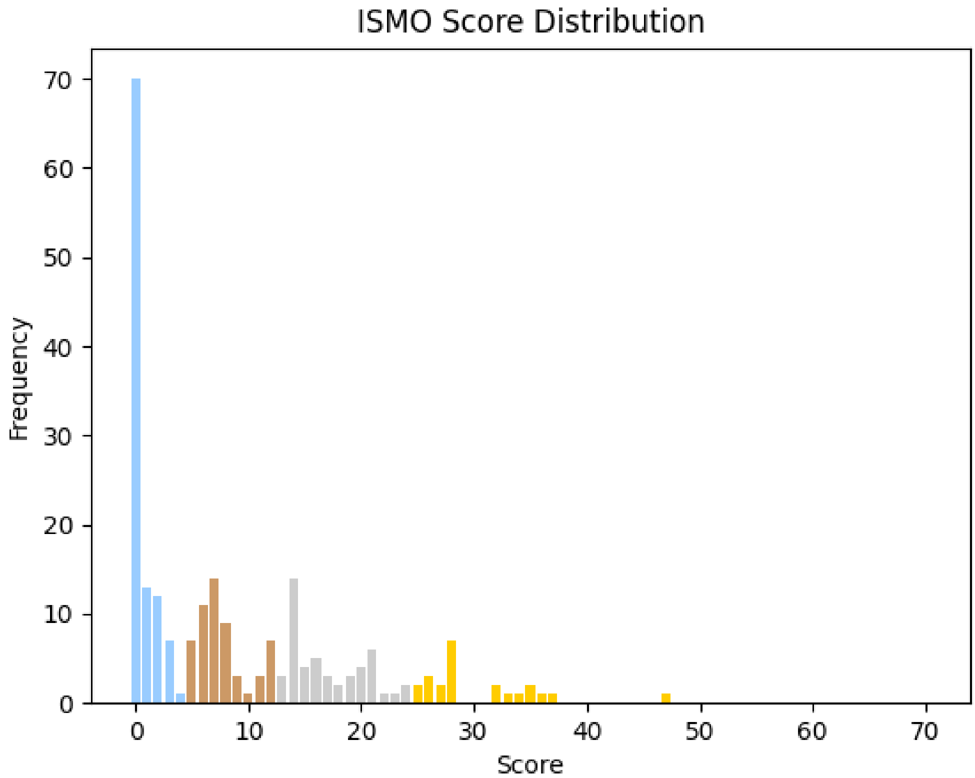







Score Distribution for Q7
Score Distribution for Q8
Score Distribution for Q9
Score Distribution for Q10
Frequency
Score
0
25
50
75
100
125
150
175
200
1
2
3
4
5
6
7

2.2 Overall Score Distribution



2.3 Statistical Data

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	total
Mean	2.14	0.69	2.00	1.97	1.18	0.24	0.01	0.48	0.07	0.00	8.75
Median	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	6.00
Mode	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

3 Solutions

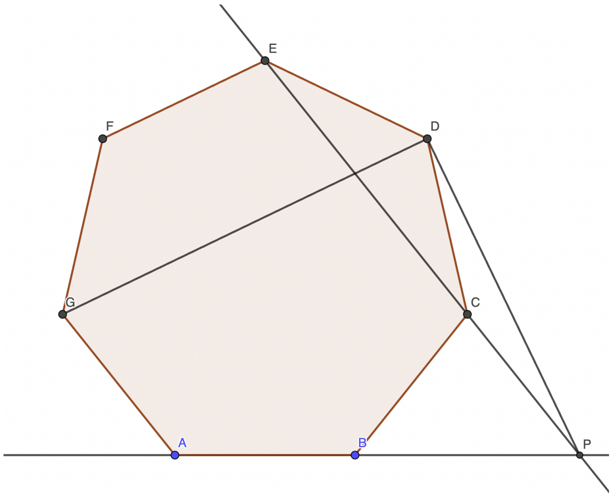
Problem 1. *(Proposed by James Xu): Find all primes that leave prime remainders when divided by all other primes.*

Solution: (by Eric Liang): Note that 0 and 1 are both not prime and also that any number when divided by 2 leaves the remainder 0 or 1. Thus, the only prime that can satisfy these conditions is 2. But also note that since every prime ≥ 2*, if we choose arbitrary prime* $p \neq 2$*, then the remainder when 2 is divided by* p *will just be 2 which is a prime. Thus, 2 is our answer.*

Marker's Report: Most people in the contest did attempt this problem, after trying small cases, most of them were able to conjecture that 2 was the only solution. However some thought that there were no solutions, which is probably a result of not reading the question properly or forgetting that 2 is an even prime and eliminating it in an argument that only works on odd primes. Those who found the correct answer but did not fully solve the problem tended to eliminate all other primes but forgot to prove that 2 was actually a solution. That being said, the process of checking 2 is actually a solution is pretty simple and thus the majority of points were awarded for eliminating all the odd primes. Furthermore, a mark was awarded for explicitly concluding that 2 was the only solution, which resulted in many contestants receiving 6/7.

Problem 2. *(Proposed by James Xu): In regular heptagon $ABCDEFG$, AB and CE intersect at P. Find the value of $\angle PDG$.*

Solution: (by James Xu): We'll first show that $BC = CP$*:* $\angle CBP = \frac{360}{7}^{\circ}$ *since it's an external angle of a heptagon. Then,* $\angle CPB = \angle ECB - \angle CBP = \angle CDE - \angle DCE - \frac{360}{7}^{\circ} = \frac{900}{7}^{\circ} - \frac{1}{2}(180 - \frac{900}{7}) :^{\circ} - \frac{360}{7}^{\circ} = \frac{360}{7}^{\circ}$



Since $\angle CBP = \angle CPB$*, we get that* $PC = CB = CD$*.*

Thus, $\angle CDP = \angle CPD = \frac{1}{2}\angle DCE = \frac{90}{7}$ *as we already calculated* $\angle DCE$ *previously*

By Symmetry, $EF//DG$*. Therefore,*

$$\angle CDG = \angle CDE - \angle GDE = \frac{900^{\circ}}{7} - \frac{360^{\circ}}{7} = \frac{540^{\circ}}{7}$$

Therefore, $\angle PDG = 90^{\circ}$ *as desired.*

Marker's Report: Many students made a good effort towards problem 2. Furthermore, many students were able to guess the answer of 90 degrees, however, due to a lack of any evidence supporting this answer they were not awarded any marks. Some students were able to identify the angle CDG *which awarded them partial marks. The majority of such students did not identifying that the triangle* DCP *was isosceles, which was a critical step, and thus did not receive full marks. Overall, although this question was well attempted by many students, very few students got full marks.*

Problem 3. *(Proposed by Michael Ma): Let a and b be real numbers satisfying*

$$(2025a - 2024b)^2 = 2025a^2 - 2024b^2$$

Prove that $a^3 + b^2 = b^3 + a^2$.

Solution: *(by Dawn Chen):* $(2025a - 2024b)^2 = 2025a^2 - 2024b^2$
$(2025a - 2024b)^2 - a^2 = 2024a^2 - 2024b^2$
$(2025a - 2024b - a)(2025a - 2024b + a) = 2024a^2 - 2024b^2$
$(2024a - 2024b)(2026a - 2024b) = 2024(a^2 - b^2)$
$2024(a - b)(2026a - 2024b) - 2024(a + b)(a - b) = 0$
$(a - b)(2026a - 2024b - a - b) = 0$
$(a - b)(2025a - 2025b) = 0$
$2025(a - b)^2 = 0$
$\therefore (a - b)^2 = 0 \Rightarrow a - b = 0$
$(a - b)(a^2 - ab + b^2) = 0 \Rightarrow a^3 - b^3 = 0$
$(a - b)(a + b) = 0 \Rightarrow a^2 - b^2 = 0$
$\therefore a^3 - b^3 = a^2 - b^2$
$\Rightarrow a^3 + b^2 = b^3 + a^2$

Marker's Report: Generally, many students attempted this question possibly due to the simple nature that it is presented with. Many students were able to "bash" the question by expanding out the brackets manually and collecting all the like terms. However, of those who had attempted the question this way, some students weren't able to get to the crux in that $(a - b)^2 = 0$, which most likely led to them manipulating the equation without a clear goal in mind, wasting a lot of time during the process. One other group of students worth noting are those who had realised that $a = b$ must be true from the end point of the problem, however failed to work from the expression given forwards to provide a comprehensive proof. Despite this, the question was generally done well by most students who attempted the problem.

Problem 4. *(Proposed by Rex Chu): Integers* $1, 2, 3, \ldots, n$ *are written on a whiteboard where* $n > 2$. *Each minute, 2 random integers* a, b *on the board are chosen and replaced with* $2a - b$ *and* $2b - a$. *Let* $f(n)$ *denote the maximum number of odd integers on the board. Find* $f(n)$.

Solution: (by James Xu): Note that using the following table, we can deduce that replacing a, b *with* $2a - b, 2b - a$ *does not change the parity of the numbers on the board. Therefore, the number of odd integers is constant, namely equal to* $\lceil \frac{n}{2} \rceil$

a	*b*	*2a-b*	*2b-a*
odd	*odd*	*odd*	*odd*
odd	*even*	*even*	*odd*
even	*even*	*even*	*even*

Marker's Report: There was quite a clear split among the students on this question. Either understanding straight away and usually getting a 6/7, or writing down not much working and skipping, usually resulting a 0. The question would appear easy to those who have seen the idea of invariant. It would be recommended to all students to write down small cases and keep track of the difference of the 3 numbers to start the question.

Problem 5. *(Proposed by Michael Ma): Let I, S, M, O be positive integers such that*

$$I \times S + M \times O = 2023$$

$$I + S \times M \times O = 2024$$

Find all possible values of $I \times S \times M \times O$.

Solution: (by James Xu): Let $L = M \times O$

Then

$$\begin{cases} IS + L = 2023 & (1) \\ I + SL = 2024 & (2) \end{cases}$$

$(1) - (2) \Rightarrow$

$$(I - L)(S - 1) = -1 \tag{3}$$

Since, I, S, M, O are all positive integers, we must have L also a positive integer. Thus, $L = I + 1, S = 2$

Thus,

$$IS + I = 2022$$
$$\Rightarrow I = 674$$

Thus,

$$I \times S \times M \times O = I \times S \times L = 674 \times 2 \times 675 = 909900$$

Marker's Report: There were quite a few promising attempts with many people being able to combine the two equations and factorize. However many candidates missed that since I, S, M and O are all integers, there are only a finite set of cases needed to be solved (as there are only finite integer factors for any finite integer). Alternatively there were also some that missed that for a fraction to be an integer we must have that the denominator is a factor of the numerator. For those that had experience with these kind of questions before, the solution solution seemed pretty straightforward, but for the others they unfortunately hit a road block they could not overcome.

Problem 6. *(Proposed by Jay Zhao): Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x - y) = f(x) - 2xy + y^2$$

Solution: (by Eric Liang): Setting $x = 0$, $y = -x$, *and* $f(0) = c$ *gives* $f(x) = x^2 + c$. *Subbing this function back in we get* $(x-y)^2+c = x^2+c-2xy+y^2$ *meaning that* c *can be any real number.*

Marker's Report: Unfortunately it was insufficient to make the substitution x=-y without rearranging the equation to $f(x) = x^2 + f(0)$ *because it does not demonstrate understanding that* $f(x) = x^2 + c$ *is a solution of the functional equation. Some students also forgot to substitute back to check their solution. Students need to understand that guessing one solution and showing that it works is not a proof and hence yields no marks.*

Problem 7. *(Proposed by Jay Zhao): How many ways can a coin be flipped n times and not land on heads or tails 3 times in a row?*

Solution: (by Jay Zhao): Call a sequence of length n ideal if it does not contain heads or tails 3 times in a row. A ideal length n cannot end with three or more of the same coin, so it must end with two or one of the same coin. Let the number of ideal sequences of length n be A_n ending with two of the same coin, and let B_n be the number of ideal sequences ending in only one of the same coin

We claim that $A_n = 2F_{n-1}$ and $B_n = 2F_n$ where F_n is the nth fibonacci number with defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$

Base Case: when $n = 1$ we have the options (H), (T) $A_n = 0$ and $B_n = 2$

Inductive Hypothesis; Assuming $A_{n-1} = 2F_{n-2}$ and $B_{n-1} = 2F_{n-1}$ A length n sequence is ideal, only if the length $n-1$ sequence we get when we remove the last coin is still ideal, as otherwise there exists a sequence of 3 heads or tails somewhere in our length n sequence. So we can construct all length n ideal sequences from appending onto length $n-1$ ideal sequences. To get a n length ideal sequence ending in two of the same coin we must have a $n-1$ length sequence with one coin at the end and add that append that one type of coin. So $A_n = B_{n-1} = 2F_{n-1}$ To get a n length ideal sequence ending in one of the same coin we can either have an $n-1$ sequence ending in two of the same coin or one of the same coin, and just append a different face of the coin. So $B_n = A_{n-1} + B_{n-1} = 2(F_{n-1} + F_n - 2) = 2F_n$

Our claim is now proven by mathematical inductive. Hence the total number of ideal length n sequences is $A_n + B_n = 2(F_n + F_{n-1}) = 2F_{n+1}$)

Marker's Report: There were very little attempt for this question overall. Of those who attempted the question, many referred to 2^n as that is the number of ways to flip n coins without any restrictions. However, this was far from the truth for this problem. Very few students actually tried small cases, which in fact often resulted in solving the question or at leasting gaining partial marks.

Problem 8. *(Proposed by Jerry Dai): The number* 1 *is written on a blackboard. James and Brian play a game, on their turn, they pick a divisor* n *of* 2024! *and replace the number* x *on the blackboard by* $n + x$ *or* nx*. James goes first. Each divisor may only be picked once. If the number on the board is even at the end of the game, James wins. Otherwise, Brian wins. Determine whether James or Brian has a winning strategy.*

Solution: (by Jerry Dai, James Xu): We note that if n *is even, then* nx *and* $n + x$ *will be even if* x *is even.*

Now we prove that there are more even divisors of 2024! *than odd divisors. Since* 2024! *is even, for any odd factor* n*, we find that* $\frac{2024!}{n}$ *is an even factor. Since* 2024, 2023! *are even and can't be generated using this process, we have more even divisors than odd.*

Then James will play as follows: every turn, perform the operation nx *with an odd divisor* n *until he runs out of odd divisors. Then perform the operation* nx *with the even divisors. Note that after the odd divisors run out, James can perform* nx *to make the next* x *even. Now, by our first observation, the number must remain even since* n *is always even. Therefore, the number on the board will be even at the end of the game, meaning that James will win.*

Marker's Report: This problem was often better completed than other problems in the 7-10 range by many students. Indeed, it was a lesson to alawys write everything down, since case bashing the different ways to get an even number, showing that an even number always results in an even number being produced, could get you 1 mark. Sadly, many students found the main idea, which was to pick all the odd numbers, and did not progress from there to prove that there were more even numbers than events. Another common mistake was to think that 2024! had factors 1-2024. This is obviously not true, since most numbers from 2024! are not primes.

Problem 9. *(Proposed by Brian Zhao): Convex quadrilateral* $ABCD$ *satisfies* $AB = 9, BC = 11, CD = 6, DA = 6$*, and* BD *bisects* $\angle ABC$*. Find* $AC \times BD$*.*

Solution: (by Brian Zhao, James Xu): Let D' *be the midpoint of minor arc* AC *of the circumcircle of* ABC*. Note that* BD' *bisects* $\angle ABC$ *and* D' *lies on the perpendicular bisector of* AC*. Since* $AD = CD$*,* D *also lies on the perpendicular bisector of* AC*. However, note that since* $AB \neq BC$*, the angle bisector of* $\angle ABC$ *and the perpendicular bisector of* AC *must intersect at only one point. i.e.* $D = D'$*. Therefore,* $ABCD$ *cyclic and by Ptolemy's Theorem,* $C \times BD = 120$

Marker's Report: Unfortunately, since the late placement of the question. many people did not attempt this problem. Among those that attempted, some failed to see that ABCD is convex, and some failed to see that this is a geometry question. Among those that understood the question correctly, many tried trig bashing, with only one success. Sadly, no marks are given for partial progress of a trignometric computation. A very small minority realised that ABCD is cyclic, and only one person proved it by quoting the configuration. Some students conjectured that the answer is 120, but unfortunately the answer alone is worth nothing.

Problem 10. *(Proposed by Michael Ma): Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of positive integers. For any x_i with prime factorization $x_i = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\ldots{p_k}^{\alpha_k}$, define $x_{i+1} = (p_1+3)^{\alpha_1}(p_2+3)^{\alpha_2}\ldots(p_k+3)^{\alpha_k}$.*

Show that for all starting values of $x_1 \geq 2$, there exists a positive integer N such that $x_{i+2} \geq {x_i}^3$ for all $i \geq N$.

Solution: (by Michael Ma):

Lemma: *Let $p(x_i)$ denote largest prime dividing x_i. If $p(x_i) > 5$, then $p(x_{i+1}) :< p(x_i) :$.*

Suppose for the sake of contradiction that a prime $q \geq p(x_i) \geq 7$ divides x_{i+1}. Then it must divide some factor $r+3$ of x_{i+1}, where $r \leq p(x_i)$ is a prime factor of x_i. But $r+3$ is even and greater than 2, so q divides $(r+3)/2$, meaning $2q \leq r+3$. This is impossible since it would imply

$$2q \leq r+3 \leq q+3 < 2q$$

Hence $p(x_{i+1}) :< p(x_i)$.

Therefore, for any x_1, by the well ordering of integers, there exists some i such that $p(x_i) \leq 5$. Note that operating on any $n = 2^a3^b5^c$ gives another number of that form. This means that eventually all terms x_i can be factorised as $2^a3^b5^c$, where a, b, c are non-negative integers. Applying the recursion twice gives $x_{i+1} = 2^{b+3c}3^b5^a$ and $x_{i+2} = 2^{b+3a}3^b5^{b+3c}$. It follows that

$$\begin{aligned} x_{i+2} &= 2^b3^{-2b}5^b \times 2^{3a}3^{3b}5^{3c} \\ &= \left(\frac{2\times 5}{3^2}\right)^b \times (2^a3^b5^c)^3 \\ &= \left(\frac{10}{9}\right)^b x_i^3 \\ &\geq x_i^3 \end{aligned}$$

Marker's Report: One thing that might have kept more people from trying the problem (apart from its placement in the paper) is difficulty in understanding the question. Many people were confused by the notation and didn't understand what the question was asking. A common mistake was to think that $xi+2 = (p_1+6)^{\alpha 1}(p_2+6)^{\alpha 2}\ldots(p_k+6)^{alpha_k}$. Direct manipulation of *the algebra was not a viable approach. A handful of people tried small cases but didn't produce any substantial observations to gain partial marks; the only successful attempt was enabled by noticing the behaviour of individual primes under the recursion, especially the decomposition of large primes into smaller ones.*