Inter-School Mathematical Olympiad

ISMO Summary 2023

1 Results

1.1 Top Scorers

Senior:

1. Antony He (ACG Parnell)
2. Ubeen Sim (Westlake Boy's High School)
3. Allen Li (Rangitoto College)

Junior:

1. Daniel Xian (St Kentigern's College)
2. Oscar Prestidge (St Kentigern's College)
3. Hanson Fang (Kristin School)

Year 7-8:

1. Sophie Xu (Kristin School)
2. David Dong (Kristin School), Katie Gao (ACG Parnell)

1.2 Medal List

First name	Last Name	School	Division	Prize
John	Cai	Rangitoto	Senior	Gold
Hanson	Fang	Kristin	Junior	Gold
Rémi	Geron	St Kentigern's College	Junior	Gold
Junyi	Guo	Kristin	Junior	Gold
Antony	He	ACG Parnell	Senior	Gold
Matthew	Kim	Macleans	Junior	Gold
Allen	Li	Rangitoto	Senior	Gold
Jason (Yiming)	Luo	ACG Parnell	Junior	Gold
Richard	Meng	Westlake Boys	Senior	Gold
Oscar	Prestidge	St Kentigern's College	Junior	Gold
Ubeen	Sim	Westlake Boys	Senior	Gold
Victor	Tai	ACG Parnell	Junior	Gold

Daniel	Xian	St Kentigern's College	Junior	Gold
Sophie	Xu	Kristin	Year 7-8	Gold
Eason	Ye	ACG Parnell	Junior	Gold
Celine	Yuan	St Cuthbert's College	Junior	Gold
Jens	Zhao	St Kentigern's College	Junior	Gold
Alan	Bailey	Westlake Boys	Senior	Silver
Eileena	Bao	Rangitoto	Senior	Silver
Alina	Chen	St Cuthbert's College	Junior	Silver
Alan	Chen	St Kentigern's College	Junior	Silver
Frank	Deng	Kristin	Junior	Silver
Jason	He	Kristin	Junior	Silver
Ashley	He	St Cuthbert's College	Senior	Silver
Oscar	Horne	Wellington College	Senior	Silver
Anthony	Hu	Macleans	Junior	Silver
Alston	Huang	Rangitoto	Junior	Silver
Jay Eunje	Hwang	Macleans	Junior	Silver
Ray	Ishihara	Westlake Boys	Senior	Silver
Jacob	Johnston	Kristin	Junior	Silver
Joshua	Langford	Wellington College	Senior	Silver
Matthew	Lee	Kristin	Junior	Silver
Yu Jun Michael	Lin	Wellington College	Senior	Silver
Eric	Liu	Macleans	Junior	Silver
Richie	Liu	Kristin	Junior	Silver
Felix	Luo	Rangitoto	Junior	Silver
Benji	Mason	Wellington College	Junior	Silver
Isabelle	Ning	Kristin	Junior	Silver
Minjae	Park	Westlake Boys	Senior	Silver
Devin	Shen	Westlake Boys	Senior	Silver
Ellie	Siu	St Cuthbert's College	Junior	Silver
Chenyou	Song	Macleans	Junior	Silver
Chen	Sun	St Kentigern's College	Junior	Silver
Haowen	Xie	Macleans	Junior	Silver
Jackie	Xu	St Cuthbert's College	Junior	Silver
Owen	Yan	Kristin	Junior	Silver
Ningyuan	Yang	Wellington College	Senior	Silver
Tony	Yu	Macleans	Junior	Silver
Sophia	Zhou	ACG Parnell	Senior	Silver
Benjamin	Chan	Macleans	Junior	Bronze
Emily	Cheng	St Cuthbert's College	Junior	Bronze
Jonathan	Chia	St Kentigern's College	Senior	Bronze
Josephine	Chong	Macleans	Junior	Bronze
Oren	Dabbach	Pinehurst	Junior	Bronze
Eason	Dong	Pinehurst	Senior	Bronze

Ryan	Fan	Kristin	Junior	Bronze
Daniel	Fu	Rangitoto	Junior	Bronze
Ryan	Gao	Kristin	Junior	Bronze
Paul	Gao	Macleans	Junior	Bronze
Ethan	Gee	Wellington College	Senior	Bronze
Raymond	Guo	Wellington College	Junior	Bronze
Eason	Guo	Kristin	Junior	Bronze
Yuan (Tony)	Guo	Kristin	Junior	Bronze
Jessie	He	Pinehurst	Senior	Bronze
XinYue (Aria)	Hua	Rangitoto	Junior	Bronze
Leo	Huang	Pinehurst	Senior	Bronze
Kevin	Jeong	Pinehurst	Senior	Bronze
Vihaan	Kapoor	Wellington College	Junior	Bronze
Jiham	Kim	St Cuthbert's College	Senior	Bronze
Jervis	Lai	Macleans	Junior	Bronze
Jeremy	Lau	Westlake Boys	Senior	Bronze
Michelle	Lee	Rangitoto	Senior	Bronze
Daniel	Lee	Macleans	Junior	Bronze
Mikey	Li	Kristin	Junior	Bronze
Jerry	Li	Pinehurst	Junior	Bronze
Jung Hun	Lim	Westlake Boys	Junior	Bronze
Kelly	Liu	Pinehurst	Junior	Bronze
Vedant	M	Wellington College	Junior	Bronze
Anthony	Meng	Westlake Boys	Junior	Bronze
Josiah	Ong	ACG Parnell	Junior	Bronze
Yixin	Pan	Macleans	Junior	Bronze
Nathan	Rao	Macleans	Junior	Bronze
Jeremy	Song	ACG Parnell	Junior	Bronze
Chenti	Song	Macleans	Junior	Bronze
Alicia	Stowers	Kristin	Senior	Bronze
Aiden	Tang	Macleans	Junior	Bronze
Jason	Tao	Westlake Boys	Junior	Bronze
Richard	Tao	Macleans	Junior	Bronze
Eassin	Wang	St Cuthbert's College	Junior	Bronze
Isla	Wang	Macleans	Junior	Bronze
Owen	Wang	Pinehurst	Junior	Bronze
Sophia	Wang	Pinehurst	Junior	Bronze
Bonnie	Wang	St Cuthbert's College	Junior	Bronze
Aaron	Woodmore	Wellington College	Senior	Bronze
Devin	Xu	Pinehurst	Junior	Bronze
Daniel	Yu	Pinehurst	Junior	Bronze
Leon	Yuan	Macleans	Junior	Bronze
Bruce	Zhang	Macleans	Junior	Bronze

Vicky	Zhou	St Cuthbert's College	Senior	Bronze
Edward	Zhu	Wellington College	Junior	Bronze

1.3 Award Details

Gold: 17 Students, cut off = 25 points
Silver: 32 Students, cut off = 13 points
Bronze: 51 Students, cut off = 5 points

1.4 School Scores[1 2]

1. St Kentigern's College (G,G,G,G,S, 162 points)
2. ACG Parnell (G,G,G,G,B, 140 points)
3. Westlake Boy's High School (G,G,S,S,S, 128 points)
4. Kristin (G,G,G,S,S, 125 points)
5. Rangitoto College (G,G,S,S,S, 124 points)
6. Macleans (G,S,S,S,S, 101 points)
7. St Cuthbert's College (G,S,S,S,B, 96 points)
8. Wellington College (S,S,S,S,S, 90 points)
9. Pinehurst (B,B,B,B,B, 55 points)

[1] School scores are the sum scores of the top 5 scoring students in each school, G,S,B stands for Gold, Silver, Bronze prizes

[2] King's is not ranked due to its insufficient entrance, AGS is entered unofficially and thus no prizes or ranking are given

2 Solutions

1. Problem statement: Dawn is playing a game with numbers on a whiteboard. She writes the number 20 23 times on the board initially, and at each turn, she chooses two numbers on the whiteboard, a and b, and replaces them with the number $a+b-n$ for some constant n. The game ends when there is only one number left on the board. Find this number in terms of n.

   Solution: Denote the sum of all numbers on the board by S. Initially, $S = 460$. Every turn, the sum S reduces by n. Since the number of numbers also reduce by 1 each turn, there must have been 22 turns. Thus, the sum S when there is only 1 number is $460-22n$. Since there is only 1 number, the sum must be the number.

   Marker's Report: Of those students who attempted the problem, many were able to earn partial or full marks through simple observations or claims. However, many students did not attempt the problem, perhaps due to the intimidating wordiness of the question. For students who reached the correct answer, many did not earn full marks due to reliance on specific case(s) to compute the answer and subsequent failure to explain why their answer must be generally true (regardless of the order in which Dawn chooses the numbers on the blackboard) – a reference or an allusion to the commutative property of addition and a brief explanation would have been sufficient.

2. If the numbers $p, p+2$, and $p+4$ are all prime, show that $p+6$ is not prime.

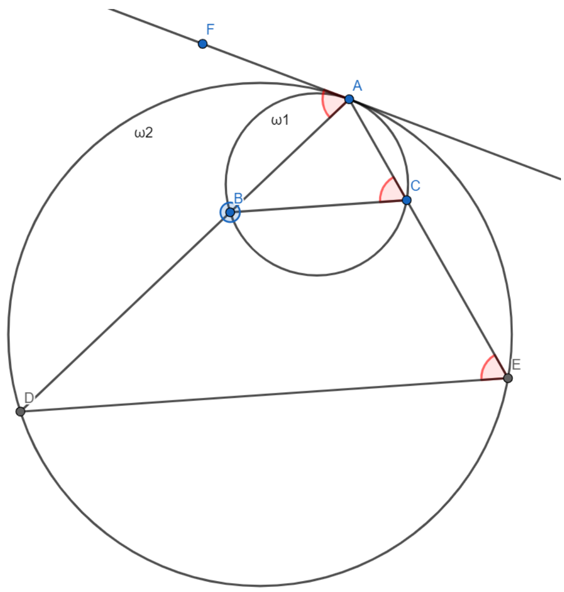
Solution: Consider $p \pmod 3$. $p+2$ and $p+4$ are equal to $p-1 \pmod 3$ and $p+1 \pmod 3$ respectively. Therefore one of p, $p+2$, and $p+4$ are divisible by 3 as it covers all 3 residue classes modulo 3. Note that unless $p=2$ or $p=3$, p, $p+2$, and $p+4$ are all greater than 3, meaning that one of them is divisible by 3 while not being 3, meaning that it is not prime. We now have the following cases:

$$\begin{cases} \text{If } p = 2, p+2 = 4 \text{ is not prime.} \\ \text{If } p = 3, p+2 = 5 \text{ and } p+4 = 7. \text{ However, } p+6 = 3+6 = 9 \text{ which is not prime.} \end{cases}$$

Marker's Report: Most students attempted this problem but few made substantial progress. The majority of students were able to come up with the case of $p = 3$ however lacked any type of explanation so were not rewarded any marks. Another common approach students made was considering $p, p+2, p+4 \pmod 5$ when p was odd, claiming that at least one of these numbers had to be divisible by 5 yet failed to recognise that this did not work for when p had unit digit 7 or 9. Students awarded with full marks had to consider $p, p+2, p+4 \pmod 3$ then list out all cases to prove that the only solution for p was $p = 3$.

3. Let ABC be a triangle with circumcircle ω_1. Let ω_2 be a second circle that is internally tangent to ω_1 at A. Let the intersection of lines AB and AC with ω_2 be D and E respectively. Prove that BC is parallel to DE.

Solution 1:



Let l_1 be the line tangent to ω_1 and ω_2 at A. Let F be a point on the line l_1 such that F and C are on opposite sides of the line AB.

$$\angle ACB = \angle FAB \text{ (Alternate segment theorem)}$$

$$= \angle FAD \text{ (ABD is collinear)}$$

$$= \angle AED \text{ (Alternate segment theorem)}$$

Thus as $\angle ACB$ and $\angle AED$ are corresponding angles, BC is parallel to DE as required.

Solution 2: Consider the homothety sending circle ABC to circle ADE centered at A. It must also send B to D and C to E. Thus, $BC//DE$

Marker's Report: Most students made little progress. The main mistake was the lack of geometric knowledge, most people failed to draw the graph correctly. Most students failed to understand the meaning of "internally tangent", they didn't draw a circle inside a circle. Students must have shown their working clearly without steps missing to get full marks.

4. Find the smallest positive integer a such that $\sqrt{a^2 - 2023}$ is an integer.

Solution: Suppose that $a^2 - 2023 = c^2$ for some positive integer c. Then, the equation rearranges to $a^2 - c^2 = 2023$. This factorises to $(a+c)(a-c) = 2023$. Since a, c are positive integers, we need to find the minimum sum of a pair of positive integers that multiplies to 2023. The average of that sum is a. The possible pairs are $(1, 2023), (7, 289), (17, 119)$. In particular, the 3rd pair gives the smallest average ($a = \frac{136}{2} = 68$)

Marker's Report: This problem was attempted by most students and a reasonable number of students were able to be awarded 7 marks by using the intended solution of reaching $(a+b)(a-b) = 2023$ and finding the pair of factors which gave the minimum a. However, a more significant portion of students attempted at using a case bash, of which many were incomplete or incorrect and unfortunately awarded 0 marks. For example, many students proved that a must be ≥ 45 but either directly reached the conclusion that $a = 68$ or incorrectly claimed that $a = 45$ was the solution.

5. Brian and James are playing a game. Initially there are n coins in a pile. Every turn, one of the players (alternating) can remove 3^k (where k is a non-negative integer) coins from the pile. The person who removes the last coin loses. Assuming Brian goes first, for which n can Brian guarantee his victory?

Solution: Note that each turn, an odd number of coins is removed, and that Brian is always the one who makes the odd number turns. If there are an odd number of coins then Brian will always be given an odd number pile and give James an even number pile so he loses. However, if at the start there are an even number of coins then Brian will always be given an even pile and will always give James an odd pile so Brian wins. Thus, the answer is all even n.

Marker's Report: Though many people had attempted this problem, the amount of people who had gotten a full solution worth 7 marks is not much. On average, the marks awarded for question five is sometimes only half that of another question.

Many competitors fell short of the correct solution because they had rushed too soon into the actual proof. Had they tried out a few more small cases, perhaps they would realise that all even numbers are solutions. This leads them to erroneous answers which they try to prove, but unfortunately without making much progress.

6. Suppose $n = 2^6 \times 3^9 \times 5^{42}$. How many positive divisors of n^2 are not divisors of n?

Solution: Note that $n^2 = 2^{12} \times 3^{18} \times 5^{84}$. Thus the number of divisors of n^2 is $13 \times 19 \times 85 = 20995$, and the number of divisors of n is $7 \times 10 \times 43 = 3010$. Thus, the number of positive divisors of n^2 that are not divisors of n is $20995 - 3010 = 17985$ since all divisors of n are divisors of n^2

Marker's Report: Although arguably one of the easier problems in the exam, there were relatively few students who attempted this problem. It may have been because the numbers chosen for the question were too intimidating. Out of those that did try, many got to the step $n^2 = 2^{12} \times 3^{18} \times 5^{84}$ but were unable to proceed. The crucial step here was to recognize that the number of factors of n and n^2 are given by the expressions $(6+1)(9+1)(42+1)$ and $(12+1)(18+1)(84+1)$ respectively. Students that achieved full marks realized that any factor of n must also be a factor of n^2 and thus arrived at the correct answer of 17985. However, many students that got this answer failed to explain why all factors of n are also factors of n^2, which, although trivial, had to be stated to achieve the full 7 marks in this question.

7. The sequence $a_1, a_2, a_3 \ldots$ is defined by $a_1 = 5$ and $a_2 = 11$, and $a_{n+2} = 2a_{n+1} - 3a_n$ for all integers $n \geq 1$. Prove that a_{2022} is divisible by 11.

Solution 1: We will show by induction that the sequence is periodic with length 5 (mod 11).
Base case: a_1, a_2, a_3, a_4, a_5 are calculated to be 5, 11, 7, -19, -59 respectively, which are 5,0,7,3,7 (mod 11).
Inductive step: Assume that the claim is true for a_{5k-4}, a_{5k-3}, a_{5k-2}, a_{5k-1}, and a_{5k} being congruent to 5,0,7,3,7 (mod 11) respectively. Then

$$\begin{aligned}
a_{5k+1} &\equiv 2 \times 7 - 3 \times 3 = 5 \pmod{11} \\
a_{5k+2} &\equiv 2 \times 5 - 3 \times 7 = -11 \equiv 0 \pmod{11} \\
a_{5k+3} &\equiv 2 \times 0 - 3 \times 5 = -15 \equiv 7 \pmod{11} \\
a_{5k+4} &\equiv 2 \times 7 - 3 \times 0 = 14 \equiv 3 \pmod{11} \\
a_{5k+5} &\equiv 2 \times 3 - 3 \times 7 = -15 \equiv 7 \pmod{11}
\end{aligned}$$

which completes the inductive step; hence the claim is proven.
Since $2022 = 5 \times 404 + 2$, then a_{2022} is congruent to 0 (mod 11), or a multiple of 11.

Solution 2: Consider the characteristic polynomial $\chi(x)$ of the sequence: $\chi(x) = x^2 - 2x + 3$. This polynomial is equivelant to $x^2 - 2x - 8$ modulo 11, which has roots 4 and -2 so $a_n \equiv A4^n + B(-2)^n \pmod{11}$. Solving for integers A, B such that the congruence holds, we get $4A - 2B \equiv 5 \pmod{11}$ and $5A + 4B \equiv 0 \pmod{11}$. Thus $A = 5, B = 2$. So $a_{2022} \equiv 5 \times 4^{2022} + 2 \times 2^{2022} \equiv 5 \times 4^2 + 2 \times 2^2 = 8 \equiv 0$ by Fermat's Little Theorem.

Marker's Report: Most students made little progress, either because of the technical phrasing or the placing of this question in the exam. Some were able to gain partial marks did so by noticing a pattern from calculating the first few terms of the sequence correctly (e.g. terms in the form of a_{5k+2} are 11). However, it is not enough to just point out that the sequence repeats every five terms or that each term is only dependent on the previous two terms. To get full marks students had to produce a full induction, including the correct base case and inductive step.

8. Consider a point inside a unit square (side length 1) to be brilliant if the sum of the 4 distances from the brilliant point to the 4 vertices of the square is $2\sqrt{2}$. Find the set of brilliant points.

Solution: Let the square be $ABCD$. Call the point P. We want to find all points where $PA + PB + PC + PD = 2\sqrt{2}$.
Consider the possibly degenerate triangles PAC and PBD. By the triangle inequality, we have

$$AC \leq PA + PC, BD \leq PB + PD$$

$$2\sqrt{2} = AC + BD \leq PA + PB + PC + PD$$

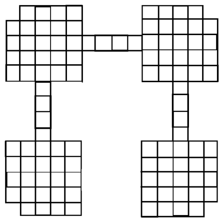
and thus

$$2\sqrt{2} \leq PA + PB + PC + PD.$$

Equality only holds if both PAC and PBD are degenerate triangles, so they must both be straight lines. Therefore P lies on both AC and BD. The intersection of AC and BD is the center the square $ABCD$. Therefore it is the only brilliant point.

Marker's Report: This question did not require any knowledge beyond the basic triangle inequality. The main difficulty was realising that the diagonals could be used as a side in the triangles, which was hinted by the answer being that the only brilliant point was at the centre of the square, which was also the intersection of the diagonals. Very few candidates solved the problem, but many got 1 mark for stating that the only brilliant point was at the centre of the triangle without proof. Then mean mark was 0.63 and the median was 0. The most successful approach was to draw in the diagonals, as well as some arbitrary point for P, and realise that using the triangle inequality on triangles PAC and PBD led to the inequality $PA + PB + PC + PD \geq AC + BD = 2\sqrt{2}$. As a result, both PAC and PBD must be degenerate triangles, leading to P lying in the centre of the square. Many candidates attempted to use coordinates to solve the question: however, most failed due to the large number of variables and radicals involved. The most and only successful solution using coordinates was considering the centre of the square and proving that any shift would result in an increase of $PA + PB + PC + PD$. Overall, this question did not require impressive formulae or theorems and instead relied on a single insight into the problem.

9. Let N be the number of ways of filling this shape with 4-block Tetris shapes (any shape consisting of 4 unit squares connected by common edges such that every two either don't intersect, intersect at a single vertex, or share an entire edge. Note that you are not playing Tetris and you can place the pieces anywhere on the grid). Show that there is an integer m such that $(m^2 - 4m + 8)(m^2 + 4m + 8) = N + 64$.



Solution: Expanding the LHS yields $m^4 + 64 = N + 64$. Thus, it is sufficient to show that $N = m^4$ for some integer m. Notice that since each of the cut 5×5 squares has exactly 24 squares, and that each tetromino covers exactly 4 squares, the 1×4 sections cannot share any tetromino with the bottom sections. Thus, the 2 vertical 1×4 sections must be filled by exactly 1 tetromino. Applying a similar argument again shows that the number of tiling of the entire shape is exactly that of tiling the 4 square sections with one corner cut independently, and thus a number to the 4th power.

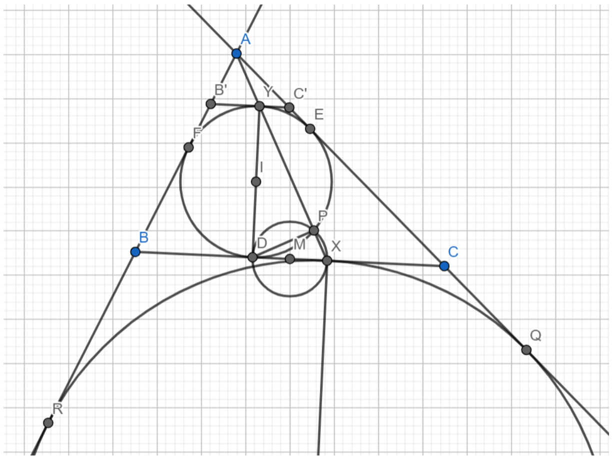
Marker's Report: Most students did not attempt the problem, as they were probably intimidated by the diagram, the algebra, and the fact that its question 9. Out of those who did attempt the problem but not the solution, most started with the algebra, likely because of their inexperience with similar problems, and somehow mostly ended in failure. Those who succeeded usually got 2 marks. very few exceptional students spotted the pattern straight away and wrote down a solution.

10. Let ABC be an acute triangle. Let I be the intersection of the internal bisectors of CAB and ABC. Let D be a point on BC such that ID is perpendicular to BC. Let M be the midpoint of BC. Let X be the reflection of D across M and Y be the reflection of D across I. Let $P \neq D$ be a point such that $MP = MD$ and $ID = IP$.

   (i) Prove that P, X, Y are colinear

   (ii) Prove that A, P, X are colinear

Solution:



(i) $ID = IP = IY$, so D, P, Y all lie on a circle with centre I and diameter DY. By angle in a semicircle theorem $\angle DPY = 90$. Similarly, $MD = MP = MX$, so D, P, X all lie on a circle with centre M and diameter DX. By angle in a semicircle theorem $\angle DPX = 90$. So $XPY = DPX + DPY = 180$. So P, X, Y colinear.
(ii) It is sufficient to prove that A, Y, X are colinear, since P lies on XY. Also notice that I is the incentre, and D is the incircle contact point on BC.

Lemma: X is the A-excircle contact point on BC.
Proof: Let E, F be the incircle contact points on AC, AB respectively. Let S, Q, R be the excircle contact points on BC, AC, AB respectively. $AE = AF, BD = BF, CD = CE, BS = BR, CS = CQ$ and $AQ = AR$ by equal tangents theorem.

$$AB + BC + CA = AE + AF + BD + BF + CD + CE = 2(AE + CE + BD)$$

$$2AQ = AR + AQ = AB + BR + AC + CQ = AB + BS + AC + CS = AB + BC + CA$$

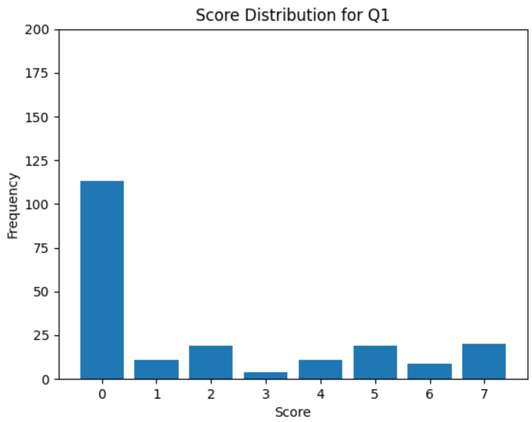
So $2AQ = 2(AC + CE + BD), AQ = AC + CE + BD, CQ = CS = BD$, so $S = X$, the reflection of D over M.
Now consider the line $B'C'$ which passes through Y and is parallel to BC. The incircle is the A-excircle of $AB'C'$ and touches $B'C'$ at Y, so the homothety centred at A sending $B'C'$ to BC would also send Y to X, meaning that A, Y, X are colinear.

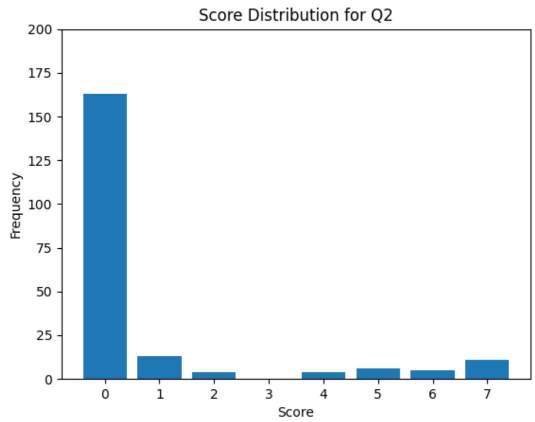
Marker's Report: Part (i) The key is realising that DPX = DPY = 90. This another instance in which drawing a large, to-scale diagram would help immensely, which is present in almost every solution we receive. Some unsuccessful (and 1 successful) attempts were made using coordinate geometry. Here is a reminder that this method usually takes a while and might not be the best use of your time. Some participants did not understand the term "colinear". Hopefully we will add its definition next year. Students should also ask questions regarding phrases they don't understand as it often occurs in Olympiad exams.

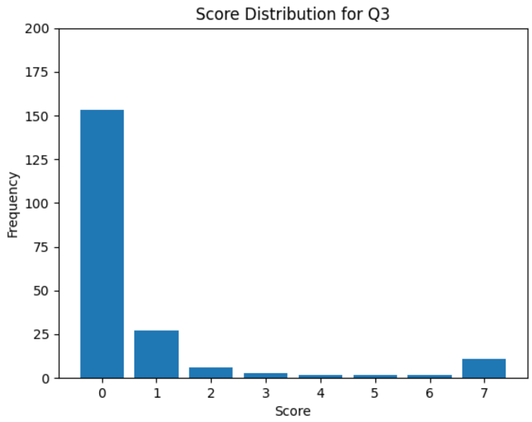
Part (ii) We only received 1 solution for this part. The key is realising that X is the A-excircle contact point.

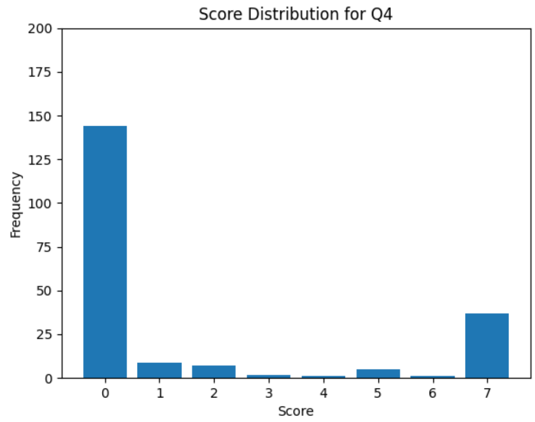
3 Score Distribution

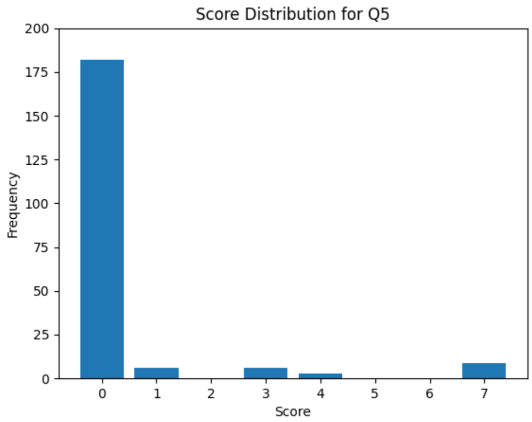
3.1 Score Distribution by Question

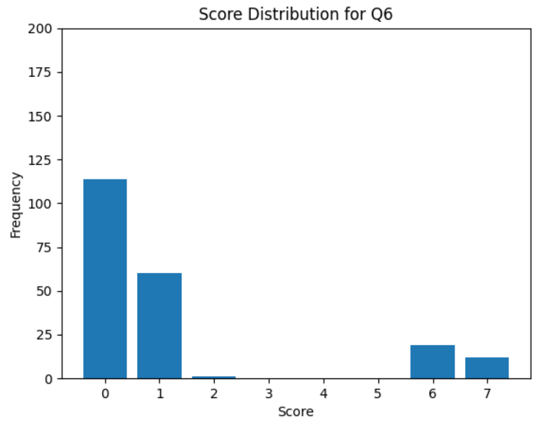


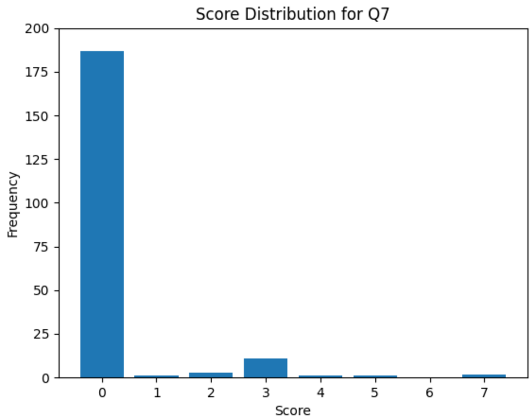


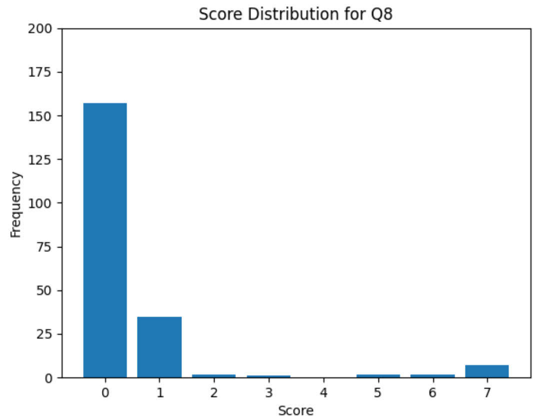


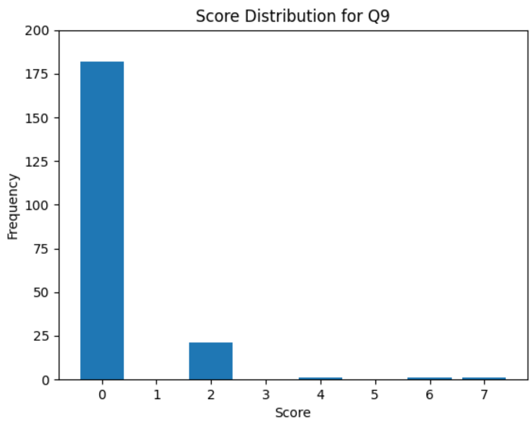


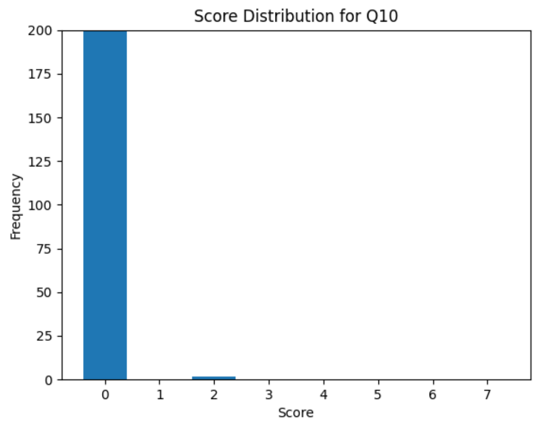




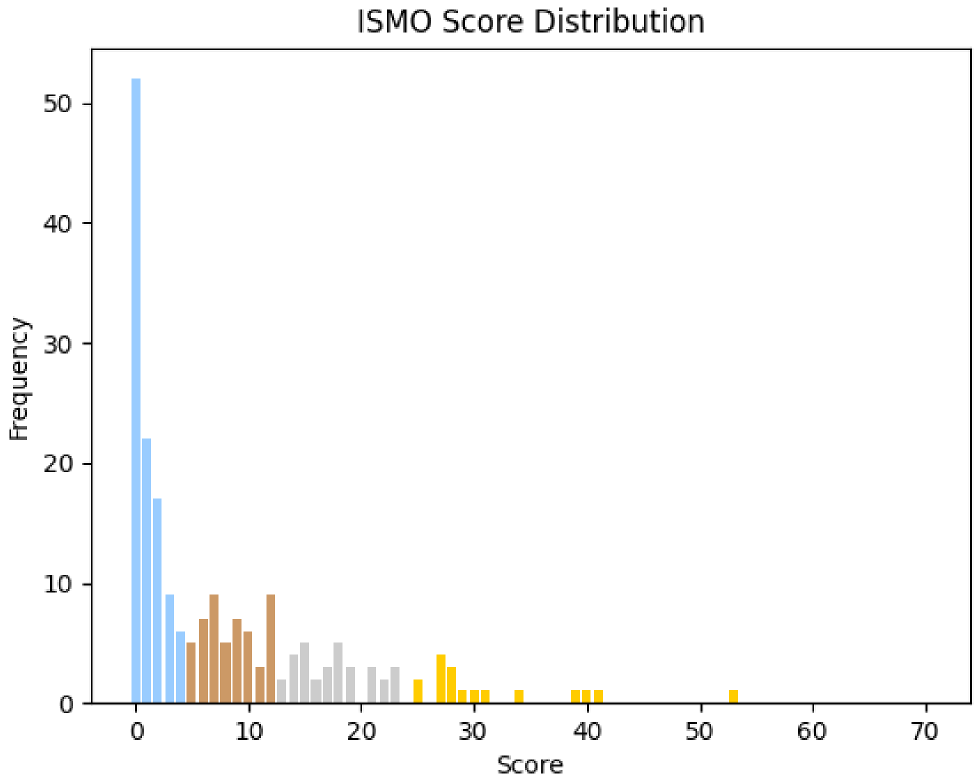

Score Distribution for Q7
Frequency
0 25 50 75 100 125 150 175 200
0 1 2 3 4 5 6 7
Score


Score Distribution for Q8
Frequency
0 25 50 75 100 125 150 175 200
0 1 2 3 4 5 6 7
Score


Score Distribution for Q9
Frequency
0 25 50 75 100 125 150 175 200
0 1 2 3 4 5 6 7
Score


Score Distribution for Q10
Frequency
0 25 50 75 100 125 150 175 200
0 1 2 3 4 5 6 7
Score

3.2 Overall Score Distribution



3.3 Statistical Data

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	total
Mean	1.91	0.84	0.75	1.57	0.48	1.26	0.31	0.55	0.29	0.02	7.98
Median	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	4.00
Mode	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

4 Distribution by School

Schools	Mean	Median	Mode
ACG	10.5	4.00	0.00
King's	2.00	2.00	2.00
Kristin	7.31	1.50	0.00
Macleans	6.98	5.50	0.00
Pinehurst	4.63	3.5	2.00
Rangitoto	7.33	2.00	0.00
St Cuthbert's	8.53	5.00	0.00
St Kentigern's	17.1	13.0	0.00
Wellington College	6.71	3.50	0.00
Westlake Boy's	10.1	6.50	0.00