2022 Inter-School Mathematical Olympiad

Marker's Report

1. General Comments

The ISMO this year was a success with more than 170 students from 6 different schools across New Zealand entering. Many students performed exceptionally well, exceeding what the problem selection committee has expected. After a round of marking and a round of moderation, the top 3 students overall are presented below:

Junior
1st Place: Kevin Xu (Pinehurst)
2nd Place: Jay Zhao (Macleans)
3rd Place: Leo Wang (Rangitoto)

Senior
1st Place: Helaman Hatcher (Burnside)
2nd Place: Jason Chen (Kristin)
3rd Place: James Wang (Burnside)

Special Prize (Question 4): Jason Chen (Kristin)

The prize cut-offs this year were:
Gold: 26
Silver: 17
Bronze: 10
Additionally, any student who received full marks on at least 1 question received an honourable mention. The full score distribution is provided at the end.

Note: Special prizes are awarded for particularly elegant solutions, as per mathematical Olympiad convention.

2. Prize Winners

ACG Parnell	Junior	Antony He	Gold
Macleans	Senior	Eric Deng	Gold
Burnside	Senior	Helaman Hatcher	Gold, Top Scorer
Macleans	Senior	James Hui	Gold
Burnside	Senior	James Wang	Gold
Kristin	Senior	Jason Chen	Gold, Special Prize (Q4)
Macleans	Junior	Jay Zhao	Gold
Rangitoto	Junior	John Cai	Gold
Pinehurst	Junior	Kevin Xu	Gold
Rangitoto	Junior	Leo Wang	Gold
Macleans	Senior	Michael (Junyou) Li	Gold
Macleans	Junior	Raymond Zhang	Gold
Macleans	Senior	William Gao	Gold
Kristin	Junior	Alicia Stowers	Silver
Rangitoto	Junior	Allen Li	Silver
Macleans	Junior	Alston Yam	Silver
Macleans	Junior	Andrew Chen	Silver
Macleans	Junior	Arshia Lotfi	Silver
ACG Parnell	Junior	Chenghao Li	Silver
Macleans	Junior	Chenyou Song	Silver
Burnside	Senior	Devin Lin	Silver
Rangitoto	Senior	Esther Liu	Silver
Kristin	Junior	Jack Shaw	Silver
Kristin	Junior	Jerry Xuan	Silver
Macleans	Senior	Joshua Yee	Silver
Kristin	Junior	Junyi Guo	Silver
Pinehurst	Junior	Kenneth Wang	Silver
Burnside	Senior	Misha Pavlov	Silver
Burnside	Senior	Nick Zheng	Silver
Kristin	Junior	Nicole Wong	Silver
Macleans	Senior	Penny Tang	Silver
Burnside	Senior	Ritvik Sharma	Silver
Rangitoto	Senior	Ruixi Liu	Silver
Burnside	Junior	Ryan Li	Silver
Kristin	Senior	Sarah Li	Silver
Kristin	Senior	Selina Ren	Silver
Kristin	Senior	Selwyn Liu	Silver
Rangitoto	Senior	Sophie Wu	Silver
Macleans	Senior	Valerie Lau	Silver
Macleans	Junior	Wesley Lau	Silver
Rangitoto	Senior	William Wei	Silver
Rangitoto	Junior	Yeonsu Na	Silver
Macleans	Junior	Yixue Wang	Silver
Kristin	Senior	Alan Qin	Bronze
Rangitoto	Junior	Alan Sun	Bronze
Macleans	Junior	Alice Zhang	Bronze
Burnside	Senior	Ameya Raut	Bronze
Macleans	Junior	Amy Zuo	Bronze
ACG Parnell	Junior	Aran Chen	Bronze

Pinehurst	Junior	Bobby Zhang	Bronze
Kristin	Junior	Byron Gao	Bronze
Pinehurst	Junior	Cynthia Zhang	Bronze
Macleans	Junior	David Guan	Bronze
Rangitoto	Senior	Echo Dong	Bronze
Macleans	Senior	Eddy Zheng	Bronze
ACG Parnell	Junior	Eric Liu	Bronze
Burnside	Junior	Erin Yamada	Bronze
Macleans	Junior	Gisele Chong	Bronze
Kristin	Year 7-8	Hanson Fang	Bronze
Macleans	Junior	Hayden Tong-Ho	Bronze
Pinehurst	Junior	Hermione Xie	Bronze
Kristin	Junior	Jacob Johnston	Bronze
Pinehurst	Junior	Jessie He	Bronze
Rangitoto	Senior	Jonathon Siah	Bronze
Rangitoto	Senior	Jonathon Sun	Bronze
Macleans	Junior	Justin Huang	Bronze
Pinehurst	Junior	Kevin Guo	Bronze
Pinehurst	Junior	Leo Huang	Bronze
Kristin	Junior	Matthew Lee	Bronze
ACG Parnell	Junior	Nicholas Chen	Bronze
Pinehurst	Junior	Oren Dabbach	Bronze
Kristin	Junior	Richie Liu	Bronze
Burnside	Junior	Ron Livne	Bronze
Kristin	Junior	Ryan Gao	Bronze
Kristin	Junior	William Huang	Bronze
Pinehurst	Junior	Alex Liu	Honourable Mention
Macleans	Senior	Andy Qin	Honourable Mention
Kristin	Senior	Belle Li	Honourable Mention
Macleans	Senior	Benny Tan	Honourable Mention
Rangitoto	Junior	Cael Sia	Honourable Mention
Macleans	Senior	Eason Chang	Honourable Mention
ACG Parnell	Junior	Eason Ye	Honourable Mention
Kristin	Junior	Frank Deng	Honourable Mention
ACG Parnell	Junior	Franklin Yuan	Honourable Mention
Kristin	Year 7-8	Isabelle Ning	Honourable Mention
Rangitoto	Junior	Judy Hu	Honourable Mention
Macleans	Junior	Leon Yuan	Honourable Mention
Kristin	Junior	Mikey Li	Honourable Mention
Macleans	Junior	TJ Kichavadi	Honourable Mention
Burnside	Junior	Ysabella Ho	Honourable Mention
Burnside	Junior	Zoey Kenix	Honourable Mention

3. Problem Specific Comments and Solutions to Problems

Please note that the mark scheme provided are not "full solutions." It only contains important portions of solutions and may not earn full marks without further explanation.

Q1 (Proposed by James Xu, Original): Find all solutions to the equation $((x+2)^2-7)^2 = 81$

(Solution by James Xu)
$((x+2)^2-7)^2 = 81$, so $(x+2)^2 - 7 = \pm 9$. [1 mk]
Thus, $(x+2)^2 = 16$ or -2. However, squares are non-negative, so $(x+2)^2 = 16$ [1 mk]
Therefore, we must have $x = 2$ or -6 [1 mk]
All 3 steps complete with or without minor flaw [4 mk]
NB: Penalize 1 mark for forgetting about $x = -6$, do not deduct marks if complex solutions are wrong.

Marker's Comment:
Most students were able to make substantial progress. A common mistake, however, was that some students failed to acknowledge that they must find "all solutions". For example, it is essential to address both of the following cases: $(x+2)^2 - 7 = 9$ and $(x+2)^2 - 7 = -9$, even if only one has real solutions. Few students expanded the expression into a quartic equation, but it was rarely successful.

Q2 (Proposed by Michael Ma): The flag of Nepal is unique as it is not rectangular. Instead, it is made from putting together 2 triangles. The method of making the shape is shown below

1. Construct a line segment AB
2. Construct a line segment AC, such that AC is perpendicular to AB and the ratio of length between AB and AC is $3:4$. Let D be on AC such that $AD = AB$. Connect BD.
3. Let E be a point on BD such that $BE = AB$
4. Construct FG such that it passes through E, is parallel and equal to AB, and have the point F on AC
5. Connect CG

(Solution by James Xu)
Diagram [0 mk]
$Area(ABD) = \frac{1}{2}$ [1 mk]
$DE{:}DB = \sqrt{2} - 1{:}\sqrt{2} = \frac{2-\sqrt{2}}{2}$. Thus, $Area(DEF) = \frac{6-4\sqrt{2}}{8}$ [1 mk]
$CF = \frac{1}{3} + \frac{2-\sqrt{2}}{2}$. Thus $Area(CGF) = \frac{\left(\frac{1}{3}+\frac{2-\sqrt{2}}{2}\right)(1)}{2} = \frac{8-3\sqrt{2}}{12}$ [1 mk]
So, $Area(CGEBA) = \frac{1}{2} + \frac{8-3\sqrt{2}}{12} - \frac{3-2\sqrt{2}}{4} = \frac{5+3\sqrt{2}}{12} \approx 0.77$ [2 mk]
Complete [2 mk]

Alternate Solution:
Area of trapezium $ABGC = \left(\frac{4}{3} + \frac{\sqrt{2}}{2}\right) \times 1 \div 2 = \frac{8+3\sqrt{2}}{12}$ [2 mk]
Area of triangle $BEG = \frac{1^2}{4} = \frac{1}{4}$ [1 mk]
So total $Area = \frac{8+3\sqrt{2}}{12} - \frac{1}{4} = \frac{5+3\sqrt{2}}{12} \approx 0.77$

NB: Allow the same answer but given by G on the opposite side of AC. Give no credit for labelling on the diagram because a) lack of explanation b) labelling is not a formal definition.

Marker's Comment:
Most students took a correct approach to the problem and made some progress, but it was particularly surprising to see that many students made calculation mistakes and thus the problem was done particularly poorly for its difficulty. There were 3 main varieties of approaches, ranging from low to high calculation complexity. There were almost no mistakes in the low computationally difficulty method, while many of students who took a direct approach lost marks for small mistakes

Q3 (Proposed by James Xu, Original): Alex and Brian are playing a game. There is a $\frac{1}{3}$ chance that Alex wins and $\frac{2}{3}$ chance that Brian wins. Alex starts with 100 coins, and Brian with infinitely many. Every time Alex wins, Alex gets half of the amount he currently has from Brian, and loses a quarter every time he loses. Determine whether Alex should play this game or not.

(Solution by Brian Zhao)
The expected value for x coins after 1 round is $E(x) = \frac{2}{3} \times 0.75 \times x + \frac{1}{3} \times 1.5 \times x = x$. [3 mk]
Thus, the number of coins Alex would have on average remains at 100. [1 mk]
And therefore, it does not matter whether he plays or not [1 mk]
Complete [2 mk]

Marker's Comment:
Very few people properly calculated expected value. A mistake made by many (at least 2/3 of those who attempted the problem, if not more) was to assume that Brian would win exactly two out of every three rounds, which is a misunderstanding of probability, as the expected value is not the same as the result after the expected value number of wins. Those who did properly calculate the expected value of Alex's coins to be 100 after each round generally went on to obtain 7s.

Q4 (Proposed by James Xu, Original): Given that $x^2 + y = 4$, $x, y > 0$ find the maximum value of:

a) x^2y
b) xy

(Solution by James Xu)
By AM-GM, $4 = x^2 + y \geq 2\sqrt{x^2y}$ [1 mk]
Thus, $2 \geq \sqrt{x^2y}$ and $4 \geq x^2y$ [1 mk]
On the other hand, taking $x = \sqrt{2}, y = 2$ yields $x^2y = 4$. [1 mk]
b)
By AM-GM, $4 = x^2 + \frac{y}{2} + \frac{y}{2} \geq 3\sqrt[3]{\frac{x^2y^2}{4}}$ [2 mk]
Thus, $\left(\frac{4}{3}\right)^3 \times 4 > x^2y^2$, so $\frac{16\sqrt{3}}{9} > xy$ [1 mk]
On the other hand, taking $x = \sqrt{\frac{4}{3}}, y = \frac{8}{3}$ yields $xy = \frac{16\sqrt{3}}{9}$. [1 mk]
NB: Award [2 mk] if the contestant said ∞ for (b) with construction.

Marker's Comment:

1. Many candidates used calculus to solve this problem but forgot that the global maximum can occur on the endpoint as well. Even though it didn't affect the answer, candidates need to check the values $x = 0$ and $x = 2$ by substitution, or at least sketch the curves to receive full working marks.
2. Many candidates used AM-GM to prove that $x^2y \leq 4$, but forgot to show that it is possible to obtain $x^2y = 4$ by stating the solution $x^2 = y = 2$.
3. Some candidates incorrectly assumed that x and y are positive integers.
4. Trial and error is not a valid method because you can't try every real number.

Tip: When using calculus, always check the endpoints. When asked to give a maximum, always show that it is both possible to obtain the maximum and impossible to go higher.

Q5 (Proposed by Eric Lee): 17 dots with negligible size are put into a 8×8 chessboard. Show that it is always possible to choose 2 dots such that their distance is at most $2\sqrt{2}$

(Solution by James Xu)
Divide the square into 16 2×2 squares. [1 mk]
By PHP, there exists a square with at least 2 points [2 mk]
In a square, max distance is $2\sqrt{2}$ [1 mk]
Complete [3 mk]
NB: missing at least in line 2 deduct [2 mk]. Missing PHP deduct [1 mk].
Remark: It is possible to not use PHP and receive full marks. One would require proof by contradiction in this case.

Marker's Comment:
Many who attempted had the right idea in mind but weren't well trained enough to write out a complete rigorous solution, and therefore have lost partial marks. Few unfortunately did not relate the problem to the Pigeonhole Principle and were only able to achieve 1 or none partial marks.
Common mistakes include not dividing the squares rigorously and used sentences such as 'space the squares evenly' or 'worst case scenario' which weren't properly defined and does not earn marks, or have not quoting the Pigeonhole Principle properly, therefore assuming there must be exactly 2 dots in exactly 1 square, which is not correct in all cases of the problem.
There were also no statement requiring the dots to be on vertices, which is something some candidates assumed.

Q6 (Proposed by James Xu, Original): Show that for all real numbers a, b, c, $a^2 + b^2 + c^2 + \mathrm{ab} + \mathrm{bc} + \mathrm{ca} \geq 0$

(Solution by James Xu)
LHS rearranges to $\frac{1}{2}[(a + b)^2 + (b + c)^2 + (c + a)^2]$. [3 mk] Award [1 mk] for partial progress towards this rearrangement.
Since it is sum of 3 squares multiplied by a positive constant, it must ≥ 0 [1 mk]
All correct: [3 mk]
Award [0 mk] if AM-GM is used without eliminating negative case

Alternate solution:
(By some participants, Eric Liang)
Notice that $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a + b + c)^2 \geq 0$ [2 mk] Award [1 mk] for partial progress towards this rearrangement.
And that $a^2, b^2, c^2 \geq 0$ [1 mk]
Therefore, the sum $2(a^2 + b^2 + c^2 + \mathrm{ab} + \mathrm{bc} + \mathrm{ca}) \geq 0$ [1 mk]
Dividing by 2 on both sides gives the desired result [1 mk]
Complete [2 mk]

Marker's Comment:
Most students recognised that square numbers are necessarily greater than 0. However, many were unable to capitalise off that knowledge. Some students noticed that the equation given was very similar to the expansion of $(a + b + c)^2$ which got them a few if not full marks. Others tried to consider cases where either 0,1,2 or 3 of the values were negative but most failed to consider the magnitude of their values which often lead to invalid solutions.

Q7 (Proposed by Michael Ma): The pitch of a sound is determined by its frequency. Call 2 frequencies euphonious if the ratio between them is $2^x : 3^x$ or $3^x : 2^x$ for some integer x. Find the number of integer frequencies which is euphonious with $22!$.
(for any positive integer n, $n! = n \times (n-1) \times ... \times 1$)

(Solution by James Xu)
Notice that $x \in [-v_3(22!), v_2(22!)]$ are all solutions. [2 mk]
Also, note that any x outside the region will create a fraction as $22!$ Does not have enough prime factors 2 or 3. [2 mk]
Thus, we have $19 + 9 + 1 = 29$ such numbers [2 mk]
All correct [1 mk]
NB: allow for other explaination

Marker's Comment: A lot of students avoided this problem because the way it was presented made it seem challenging. Students who identified and/or explained that 22! needed to have enough factors of 2 or 3 in order for the other frequency to be an integer were able to receive partial marks. Many who made progress on the problem could not get full marks either because they made an error in the counting of prime factors or did not consider the case when $x = 0$.

Q8 (Proposed by James Xu, Original): Aaron and Betty are playing a game consisting of 3 piles of stones containing (a, b, c) where $a \geq b \geq c$ stones with Aaron going first. A move consists of choosing one of the b, c pile and moving a stone from that pile to each of the other piles. A player loses if they cannot make a valid move on their turn. Find all configurations (a, b, c) where Aaron can win.

(Solution by Brian Zhao)
Note that the amount a is irrelevant as we cannot remove from a. Thus, the game is equivalent to the following: remove 2 stones from a pile and put 1 in the other. First person who can't move loses. [1 mk]
Consider the number $m = b - c$. m either increases by 3 or decreases by 3 every move. [1 mk]
Thus, m is constant modulo 6 after 2 moves. [1 mk]
Noting that it is always possible to move if $m \equiv 2,3,4 (mod\ 6)$. Therefore, these are the only cases where Aaron can win [2 mk]
Completion[2 mk]

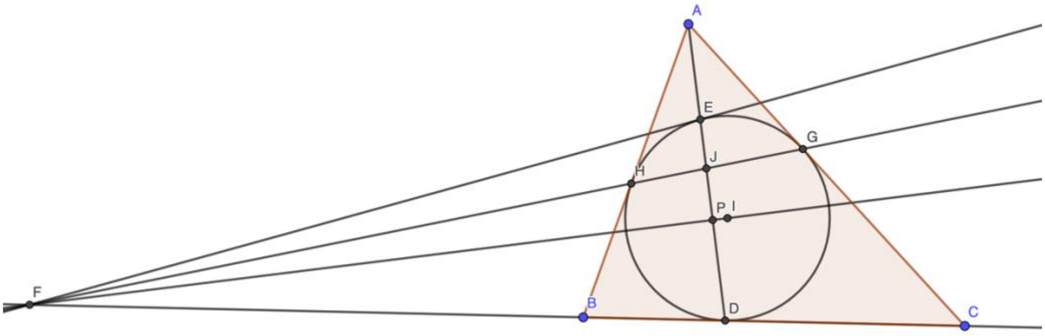
A few candidates realised that the solution is related to b-c and modulo division, but unfortunately no one considered $b - c$ in modulo 6. Other candidates mentioned that there are infinitely many solutions and stopped. Please be aware that you don't need to list all the solutions, just give a rule that fits all of them. Tip for next time: try a lot of small cases. i.e. $0 \leq c \leq 2$ and $0 \leq b \leq 11$.

Q9 (Proposed by James Xu, credit: C.Chen 2014): Find all positive integers n such that $\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}$ is an integer.

(Solution by James Xu)
By Bertrand's Postulate/Bertrand-Chebyshev Theorem, there exists a prime between n and $\frac{n}{2}$.
Thus, there must not be any solutions as the numerator will not be divisible by p after creating a common denominator. [2 mk]
This is because exactly 1 term will have the numerator not a multiple of p, namely, the term $\frac{1}{p}$.
Therefore, the only solution is 1. [1 mk]
Complete solution [3 mk]
NB: There is an alternative solution with $v_3(x)$ or $v_2(x)$, verify the solution, and award marks at the marker's discretion.
Remark: Step 1 can be replaced with Prime Number Theorem.
Remark 2: "only" is required in the answer to receive the [1 mk]

Marker's Comment:
Many students were able to gain 1 mark for guessing the result. However, the vast majority of students were unable to make further progress due to the difficulties and time limitation. Some students did take a similar approach to the official solutions and made some partial progress. One main take away from this would be to state the obvious as it may be worth marks.

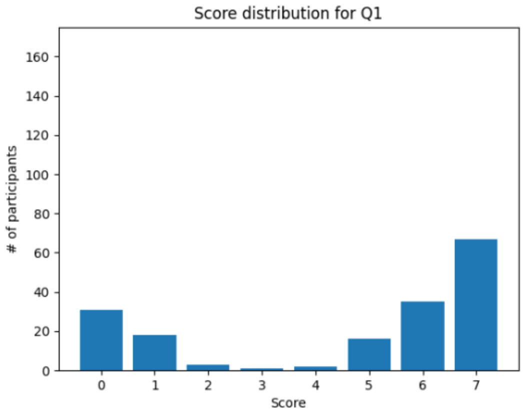
Q10 (Proposed by James Xu, Brian Zhao, credit: ELMO 2012 SL, EGMO by E. Chen 9.46):
Let ABC be a triangle with incenter I. The foot of the perpendicular from I to BC is D, and

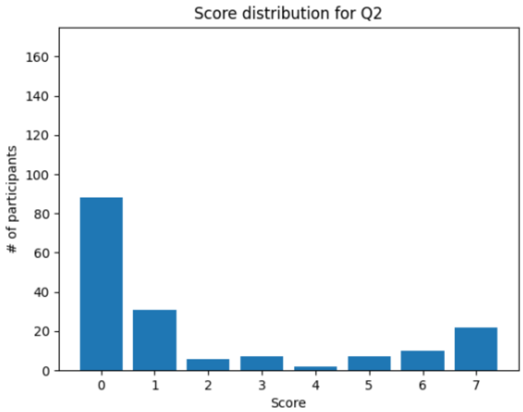


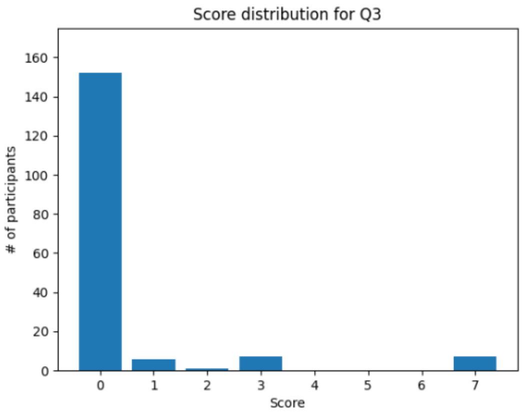
(Solution by James Xu)
Let $AD \cap$ incircle at E, $PI \cap BC = F$, The incircle is tangent to AB, AC at H, G. $HG \cap AD = J$
By symmetry, the tangent at E passes through F.[1 mk]
Note that AD is the polar of F W.R.T. the incircle. Thus, the polar of A, line HG, must pass through F[1 mk]
Consider the complete quadrilateral $HGCBAF$. We must have $(F, D; B, C) = -1$ [1 mk]
Since we are given $PI \perp PD$. Thus, by Apollonius circle, PD bisects $\angle BPC$ [1 mk]
Completion [3 mk]
Rmk:
This problem is also solvable with trigonometry, which may be more familiar to many candidates, though in a much more complex way. The key to the problem is realising the different properties the point F holds. (All 4 lines are concurrent there, as well as the harmonic bundle (F,D;B,C)).

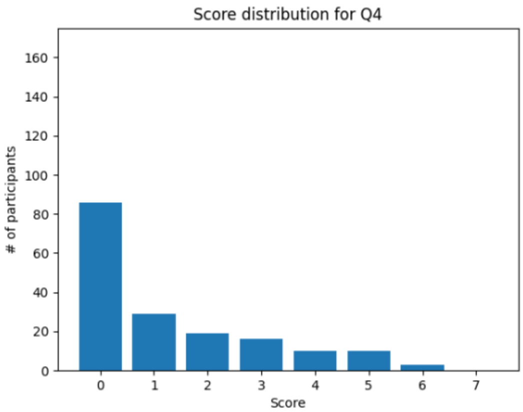
Marker's Comment:
Very little attempt was made by most students due to the difficulty of this problem and the lack of time. It was definitely the hardest problem of all and required projective geometry to make the problem easier. A trigonometric solution is also available but is much more complex. Some trivial observations were made by some students but were unfortunately mostly unhelpful in the proof of the result.

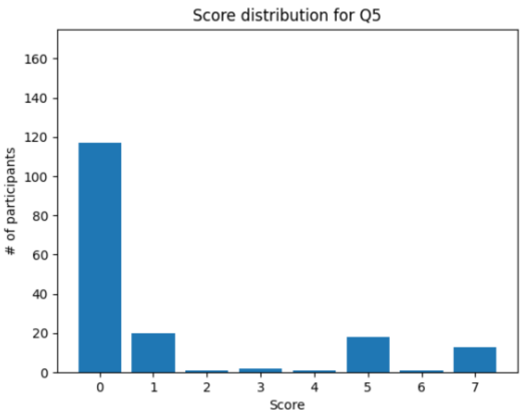
4. Score Distribution

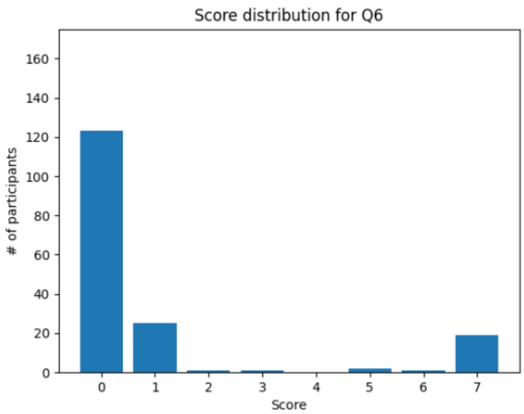


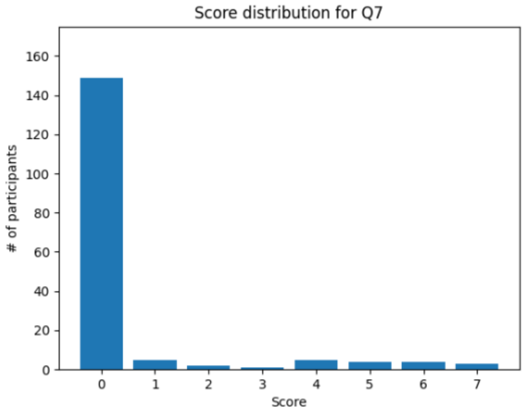











Score distribution for Q7
# of participants
Score
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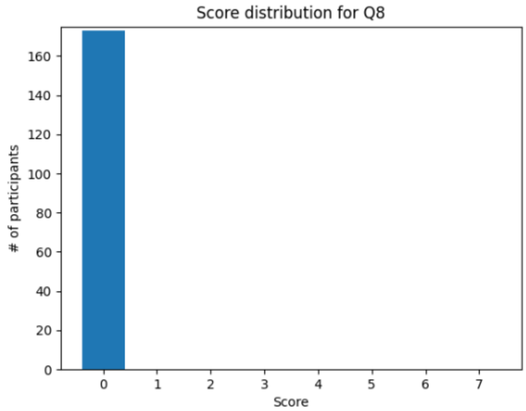
1
2
3
4
5
6
7
0
20
40
60
80
100
120
140
160


Score distribution for Q8
# of participants
Score
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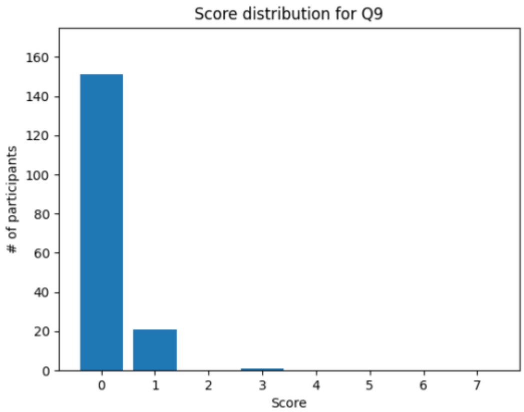
3
4
5
6
7
0
20
40
60
80
100
120
140
160


Score distribution for Q9
# of participants
Score
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6
7
0
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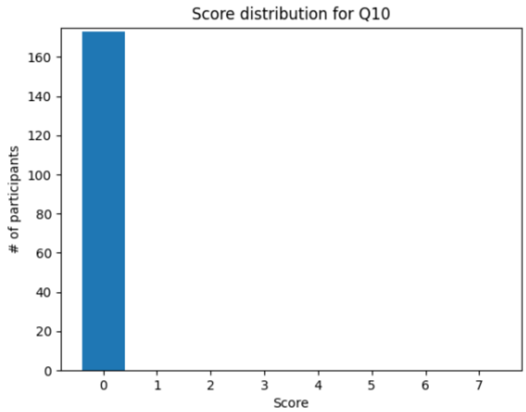
40
60
80
100
120
140
160


Score distribution for Q10
# of participants
Score
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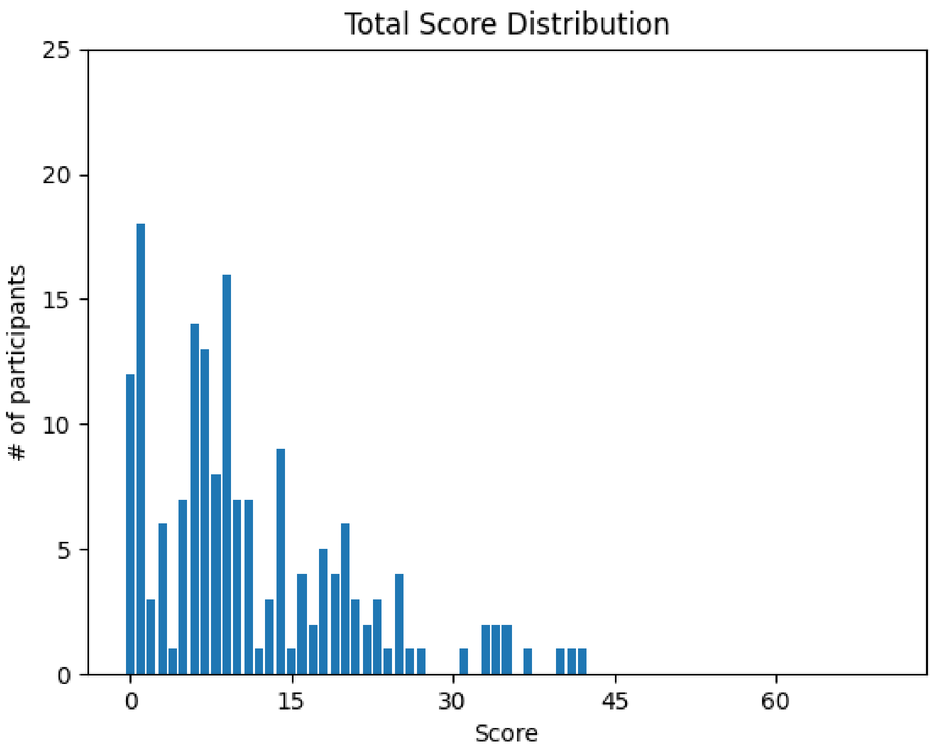
3
4
5
6
7
0
20
40
60
80
100
120
140
160


Total Score Distribution
# of participants
Score
0
15
30
45
60
0
5
10
15
20
25

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
#0	31	88	152	86	117	123	149	173	151	173
#1	18	31	6	29	20	25	5	0	21	0
#2	3	6	1	19	1	1	2	0	0	0
#3	1	7	7	16	2	1	1	0	1	0
#4	2	2	0	10	1	0	5	0	0	0
#5	16	7	0	10	18	2	4	0	0	0
#6	35	10	0	3	1	1	4	0	0	0
#7	67	22	7	0	13	19	3	0	0	0

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Total
Average	4.6	1.9	0.5	1.3	1.3	1.0	0.6	0	0.1	0	11.1849711
Highest Score	7	7	7	6	7	7	7	0	3	0	42
LQ	1	0	0	0	0	0	0	0	0	0	5
Median	6	0	0	1	0	0	0	0	0	0	9
UQ	7	3	0	2	1	1	0	0	0	0	16
Mode	7	0	0	0	0	0	0	0	0	0	1
Standard Deviation	2.8	2.6	1.5	1.7	2.3	2.3	1.6	0	0.4	0	9.495433702

Note: Due to penalties and bonuses imposed on some participants which the total column takes into account of, the data for total may not be the same as that calculated from the data of each question.